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SIMPLE STABILITY TESTS FOR SECOND ORDER DELAY DIFFERENTIAL EQUATIONS

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For linear and nonlinear delay differential equations of the second order with damping terms exponential stability and global asymptotic stability conditions are obtained. The results are based on the new sufficient stability conditions for systems of linear equations. The results are illustrated with numerical examples, and a list of relevant problems for future research is presented.

We proposed a substitution which exploits the parameters of the original model. By using that approach, a broad class of the second order non-autonomous linear equations with delays was examined and explicit easily-verifiable sufficient stability conditions were obtained. There is a natural extension of this approach to stability analysis of high-order models. For the nonlinear second order non-autonomous equations with delays we applied the linearization technique and the results obtained for linear models. Our stability tests are applicable to some milling models and to a non-autonomous Kaldor–Kalecki business cycle model. Several numerical examples illustrate the application of the stability tests. We suggest that a similar technique can be developed for higher order linear delay equations, with or without non-delay terms.

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1. Introduction

In the present paper, a specially designed substitution transforms linear second order equations into a system, to which we apply some known exponential stability results.

This and the linearization techniques are used to devise new global stability tests for nonlinear non-autonomous models. These results are explicit, easily verifiable and can be applied to a general class of second order non-autonomous equations. Some of the theorems complement results [1, 2], as well as the tests obtained in recent papers [3–5].

Consider the following system

$$\dot{x}_i(t) = -\sum_{j=1}^{m}\sum_{k=1}^{r_{ij}} a_{ij}^k(t)x_j\left(h_{ij}^k(t)\right), i = 1, \dots, m, \tag{1}$$

where $t \geq 0$, m is a natural number, $r_{ij}, i, j = 1, \dots, m$ are natural numbers, coefficients $a_{ij}^k: [0,\infty) \to \mathbb{R}$ and delays $h_{ij}^k: [0,\infty) \to \mathbb{R}$ are measurable functions.

Let $A_i, i = 1, \dots, m$ be functions defined as

$$A_i(t) := \frac{1}{a_i(t)}\left[\sum_{k=1}^{r_{ii}} a_{ii}^k(t)\int_{\max\{t_0, h_{ii}^k(t)\}}^{t}\sum_{j=1}^{m}\sum_{l=1}^{r_{ij}}|a_{ij}^l(s)|ds + \sum_{j=1, j\neq i}^{m}\sum_{k=1}^{r_{ij}}|a_{ij}^k(t)|\right],$$

where

$$a_i(t) := \sum_{k=1}^{r_{ii}} a_{ii}^k(t). \tag{2}$$

Lemma 1. *Suppose that* $a_i(t) \geq a_0 > 0, max_{i=1,\dots,m} \lim sup_{t\to\infty} A_i(t) < 1$. *Then, the system (1) is uniformly exponentially stable.*

Lemma 2. *Let* $a_{ii}^k(t) \geq 0, a_i(t) \geq a_0 > 0, i = 1, \dots, m, t \geq t_0$,

$$\max_{i=1,\dots,m} \lim_{t\to\infty} \sup \frac{1}{a_i(t)}\sum_{j=1, j\neq i}^{m}\sum_{k=1}^{r_{ij}}|a_{ij}^k(t)| < 1 \tag{3}$$

and

$$\max_{i=1,\dots,m} \lim_{t\to\infty} sup\, A_i(t) < 1 + \frac{1}{e}. \tag{4}$$

Then, the system (1) is uniformly exponentially stable.

In Lemma 1 does not assumed that $a_{ii}^k(t) \geq 0$ but in Lemma 2 the constant in right-hand side of the inequality (4) is better. So these lemmas are independent and we use both of them.

Consider first the following equation:

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) + \sum_{k=1}^{m} c_k(t)\dot{x}(h_k(t)) = 0, \quad (5)$$

where

$$0 < a \le a(t) \le A, 0 < b \le b(t) \le B, |c_k(t)| \le C_k, t - h_k(t) \le \tau_k.$$

Theorem 1. *Suppose* $\lim inf_{t\to\infty}\left(\sum_{k=1}^{m} c_k(t) + a(t) - \frac{a}{2}\right) > 0$ *and the following condition holds*

$$\limsup_{t\to\infty} \frac{1}{\sum_{k=1}^{m} c_k(t)+a(t)-\frac{a}{2}}\left[\sum_{k=1}^{m} |c_k(t)| \int_{h_k(t)}^{t} \left(\left|a(s) - \frac{a}{2} - \frac{2b(s)}{a}\right| + \right.\right.$$

$$\left.\left. 2\sum_{k=1}^{m} c_k^+(s) + a(s) - \frac{a}{2}\right) ds + \left|a(t) - \frac{a}{2} - \frac{2b(t)}{a}\right| + \sum_{k=1}^{m} |c_k(t)|\right] < 1. \quad (6)$$

Then equation (5) is exponentially stable.

Proof. By substitution to equation (5)

$$\dot{x}(t) = -\frac{a}{2}x + \left(\frac{a}{2} - \epsilon\right)y, \ddot{x}(t) = \left(\frac{a}{2} - \epsilon\right)\dot{y}(t) + \frac{a^2}{4}x - \frac{a}{2}(\frac{a}{2} - \epsilon)y$$

we have

$$\dot{y}(t) = \frac{\frac{a}{2}\left[\left(a(t)-\frac{a}{2}\right)-\frac{2b(t)}{a}\right]}{\frac{a}{2}-\epsilon}x(t) + \frac{\frac{a}{2}}{\frac{a}{2}-\epsilon}\sum_{k=1}^{m} c_k(t)x(h_k(t)) - \sum_{k=1}^{m} c_k(t)y\big(h_k(t)\big) - \left(a(t) - \frac{a}{2}\right)y(t). \quad (7)$$

By Lemma 1 the following condition implies exponential stability of system (7).

$$\lim_{t\to\infty} sup \frac{1}{\sum_{k=1}^{m} c_k(t)+a(t)-\frac{a}{2}}\left[\sum_{k=1}^{m} |c_k(t)| \int_{h_k(t)}^{t} \frac{\frac{a}{2}}{\frac{a}{2}-\epsilon}\left(\left|a(s) - \frac{a}{2} - \frac{2b(s)}{a}\right| + \right.\right.$$

$$\left.\left. +2\sum_{k=1}^{m} c_k^+(t) + a(s) - \frac{a}{2}\right) ds + \left|a(t) - \frac{a}{2} - \frac{2b(t)}{a}\right| + \sum_{k=1}^{m} |c_k(t)|\right] < 1. \quad (8)$$

But for small $\epsilon > 0$ inequality (6) implies (8). Hence system (7) and then equation (5) are exponentially stable.

Denote $c^+ = max\{c, 0\}, c^- = max\{-c, 0\}$.

Corollary 1. *Suppose* $\lim sup_{t\to\infty}\left(\sum_{k=1}^{m} c_k(t) + a(t) - \frac{a}{2}\right) > 0$ *and there exist* $t_0 \ge 0, \delta > 0$ *such that at least one of the following conditions hols:*

1. $a^2 \ge 4B$,

$$\sum_{k=1}^{m}|c_k(t)| \int_{h_k(t)}^{t}\left(2a(s) - a - \frac{2b(s)}{a} + 2\sum_{k=1}^{m} c_k^+(s)\right) ds + \sum_{k=1}^{m} c_k^-(t) < \frac{2b(t)}{a} - \delta, t \ge t_0. \quad (9)$$

2. $A^2 < 4b$,

$$\sum_{k=1}^{m}|c_k(t)| \int_{h_k(t)}^{t}\left(\frac{2b(s)}{a} + 2\sum_{k=1}^{m} c_k^+(s)\right) ds + \sum_{k=1}^{m} c_k^-(t) + \frac{2b(t)}{a} < 2a(t) - a - \delta, t \ge t_0. \quad (10)$$

Then equation (5) is exponentially stable.

Proof. Suppose conditions 1) hold. Inequality $a^2 \ge 4B$ implies that $a(s) - \frac{a}{2} - \frac{2b(s)}{a} \ge 0$. Hence (6) holds if for some $t_0 \ge 0, \delta > 0$

$$\sum_{k=1}^{m}|c_k(t)| \int_{h_k(t)}^{t}\left(a(s) - \frac{a}{2} - \frac{2b(s)}{a} + 2\sum_{k=1}^{m} c_k^+(t) + a(s) - \frac{a}{2}\right) ds + a(t) - \frac{a}{2} - \frac{2b(t)}{a} +$$

$$+\sum_{k=1}^{m}|c_k(t)| < \sum_{k=1}^{m} c_k(t) + a(t) - \frac{a}{2} - \delta, t \ge t_0. \quad (11)$$

Inequality (11) after simple transformations coincides with (9).

The second case is proved similarly.

Now with an additional assumption $c_k(t) \ge 0$ we can improve Theorem 1.

Theorem 2. *Suppose* $c_k(t) \ge 0, k = 1, \dots, m$ *and the following conditions hold:*

$$\left|a(t) - \frac{a}{2} - \frac{2b(t)}{a}\right| + \sum_{k=1}^{m} c_k(t) < 1,$$

$$\lim_{t\to\infty} sup \frac{1}{\sum_{k=1}^{m} c_k(t)+a(t)-\frac{a}{2}}\left[\sum_{k=1}^{m} c_k(t) \int_{h_k(t)}^{t}\left(\left|a(s) - \frac{a}{2} - \frac{2b(s)}{a}\right| + \right.\right.$$

$$\left.\left. + 2\sum_{k=1}^{m} c_k(s) + a(s) - \frac{a}{2}\right) ds + \left|a(t) - \frac{a}{2} - \frac{2b(t)}{a}\right| + \sum_{k=1}^{m} c_k(t)\right] < 1 + \frac{1}{e}.$$

Then equation (5) is exponentially stable.

The proof is based on Lemma 2 and is similar to the proof of Theorem 1.

Corollary 2. *Suppose*

$$a(t) \equiv a > 0, b(t) \equiv b > 0, 0 \le c_k(t) \le C_k, t - h_k(t) \le \tau_k.$$

If at least one of the following conditions hols:

1. $a^2 \geq 4b, \frac{a}{2} - \frac{2b(t)}{a} + \sum_{k=1}^m C_k < 1$ *and*
$\sum_{k=1}^m C_k \tau_k \left(a - \frac{2b}{a} + 2\sum_{k=1}^m C_k\right) < \frac{2b}{a} + \frac{1}{e}\left(\frac{a}{2} + \sum_{k=1}^m C_k\right)$.
2. $a^2 < 4b, \frac{2b(t)}{a} - \frac{a}{2} + \sum_{k=1}^m C_k < 1$ *and*
$\sum_{k=1}^m C_k \tau_k \left(\frac{2b}{a} + 2\sum_{k=1}^m C_k\right) < \left(1 + \frac{1}{2e}\right)a + \frac{1}{e}\sum_{k=1}^m C_k - \frac{2b}{a}$.
Then equation (5) is exponentially stable.

Consider here the following equation

$$\ddot{x}(t) + a(t)\dot{x}(t)) + \sum_{k=1}^m b_k(t)x(h_k(t)) = 0. \quad (12)$$

Theorem 3. *Suppose* $\lim inf_{t\to\infty}\left(a(t) - \frac{a}{2} - \sum_{k=1}^m b_k(t)(t - h_k(t))\right) > 0$ *and the following conditions holds*

$$\lim_{t\to\infty} \sup \frac{1}{a(t) - \frac{a}{2} - \sum_{k=1}^m b_k(t)(t-h_k(t))} \times$$
$$\times \left[\sum_{k=1}^m b_k(t)(t - h_k(t)) \int_{h_k(t)}^t \left(2a(s) - a - \frac{2\sum_{k=1}^m b_k(s)(s-h_k(s))}{a}\right) ds + \right.$$
$$\left. + a(t) - \frac{a}{2} + \frac{2\sum_{k=1}^m b_k(t)(t-h_k(t))}{a} + \sum_{k=1}^m b_k(t)(t - h_k(t))\right] < 1. \quad (13)$$

Then equation (12) is exponentially stable.

To prove the theorem we need in the following lemma.

Lemma 3. *Suppose* $a_k: [a,\infty) \to \mathbb{R}^+ = [0,\infty), k = 1,\dots,m$ *are measurable essentially bounded functions,* $h_k: [a,\infty) \to \mathbb{R}, h_k(t) \leq t, k = 1,\dots,m$ *are measurable functions. Then for any continuous function* $x: [a,\infty) \to \mathbb{R}$ *there exists measurable function* $h: [a,\infty) \to \mathbb{R}, h(t) \leq t$ *such that*

$$\min_k h_k(t) \leq h(t) \leq \max_k h_k(t), \sum_{k=1}^m a_k(t)x(h_k(t)) = (\sum_{k=1}^m a_k(t))x(h(t)).$$

Now we are ready to prove the theorem.

Proof. Suppose *x* is fixed solution of equation (12). Transform equation (12)

$$\ddot{x}(t) + a(t)\dot{x}(t)) + \sum_{k=1}^m b_k(t)x(t) - \sum_{k=1}^m b_k(t)\int_{h_k(t)}^t \dot{x}(s)ds = 0. \quad (14)$$

By Lemma 3 there exist $r_k(t), k = 1,\dots,m$ such that $h_k(t) \leq r_k(t) \leq t$ and

$$\int_{h_k(t)}^t \dot{x}(s)ds = (t - h_k(t))\dot{x}(r_k(t)).$$

Hence *x* is a solution of the following equation

$$\ddot{z}(t) + a(t)\dot{z}(t)) + \sum_{k=1}^m b_k(t)z(t) + \sum_{k=1}^m c_k(t)\dot{z}(r_k(t)) = 0, \quad (15)$$

where $c_k(t) = -b_k(t)(t - h_k(t)) \leq 0$ and then $c_k^+(t) = 0$. Inequality (13) implies that (6) holds, where h_k one can replace by r_k.

By Theorem 1 equation (15) is exponentially stable. Hence the solution *x* of of equation (12) tends to zero exponentially.

2. Nonlinear Equations

In this section we examine several nonlinear delay differential equations of the second order which have the following general form

$$\ddot{x}(t) + \sum_{k=1}^m f_k(t, x(p_k(t)), \dot{x}(g_k(t))) + \sum_{k=1}^l s_k(t, x(h_k(t))) = 0, \quad (16)$$

with the following initial function

$$x(t) = \varphi(t), \dot{x}(t) = \psi(t), t \leq t_0, t_0 \geq 0, \quad (17)$$

where $f_k(t, u_1, u_2), k = 1,\dots,m, s_k(t,u)$, are Caratheodory functions which are measurable in *t* and continuous in all the other arguments, condition (a2) holds for delay functions p_k, g_k, h_k; φ *and* ψ are Borel measurable bounded functions.

We will assume that the initial value problem has a unique global solution on $[t_0,\infty)$ for all nonlinear equations considered in this section.

Theorem 4. *Consider the equation*

$$\ddot{x}(t) + f(t, x(t), \dot{x}(t)) + s(t, x(t)) + \sum_{k=1}^m s_k(t, x(t), x(h_k(t))) = 0, \quad (18)$$

where

$$f(t,v,0) = 0, s(t,0) = 0, s_k(t,v,0) = 0, 0 < a_0 \leq \frac{f(t,v,u)}{u} \leq A,$$
$$0 < b_0 \leq \frac{s(t,u)}{u} \leq B, \left|\frac{s_k(t,v,u)}{u}\right| \leq C_k, u \neq 0, t - h_k(t) \leq \tau.$$

If at least one of the following conditions holds:

1) $B \leq \frac{a_0^2}{4}, \sum_{k=1}^{m} C_k < b_0 - \frac{a_0}{2}(A - a_0)$,
2) $b_0 \geq \frac{a_0}{2}\left(A - \frac{a_0}{2}\right), \sum_{k=1}^{m} C_k < \frac{a_0^2}{2} - B$,

then zero is a global attractor for all solutions of problem (18), (17).

Proof. First, by the previous theorem there exists a global solution x of problem (18), (17). Suppose x is a fixed solution of problem (18), (17). Rewrite equation (18) in the form

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) + \sum_{k=1}^{m} c_k(t)x(h_k(t)) = 0,$$

where

$$a(t) = \begin{cases} \frac{f(t,x(t),\dot{x}(t))}{\dot{x}(t)}, \dot{x}(t) \neq 0, \\ a_0, \dot{x}(t) = 0, \end{cases} \quad b(t) = \begin{cases} \frac{s(t,x(t))}{x(t)}, x(t) \neq 0, \\ b_0, x(t) = 0, \end{cases}$$

$$c_k(t) = \begin{cases} \frac{s_k(t,x(t),x(h_k(t)))}{x(h_k(t))}, x(h_k(t)) \neq 0, \\ 0, x(h_k(t)) = 0. \end{cases}$$

Hence the function x is a solution of the linear equation

$$\ddot{y}(t) + a(t)\dot{y}(t) + b(t)y(t) + \sum_{k=1}^{m} c_k(t)y(h_k(t)) = 0, \tag{19}$$

which is exponentially stable by Theorem 1. Thus for any solution y of equation (19) we have $\lim_{t\to\infty} y(t) = 0$. Since x is a solution of (19), we have $\lim_{t\to\infty} x(t) = 0$.

The previous proof is readily adapted to the proof of the following theorems.

Theorem 5. *Consider the equation*

$$\ddot{x}(t) + f\big(t, x(t), \dot{x}(t)\big) + s\big(t, x(t)\big) + \sum_{k=1}^{m} s_k(t, x(t), \dot{x}(h_k(t))) = 0, \tag{20}$$

where

$$f(t, v, 0) = 0, s(t, 0) = 0, s_k(t, v, 0) = 0, 0 < a_0 \leq \frac{f(t,v,u)}{u} \leq A,$$

$$0 < b_0 \leq \frac{s(t,u)}{u} \leq B, \left|\frac{s_k(t,v,u)}{u}\right| \leq C_k, u \neq 0, t - h_k(t) \leq \tau.$$

Suppose at least one of the following conditions holds:

1) $B \leq \frac{a_0^2}{4}, \sum_{k=1}^{m} C_k < \frac{2b_0 - a_0(A - a_0)}{2a_0}$,
2) $b_0 \geq \frac{a_0}{2}\left(A - \frac{a_0}{2}\right), \sum_{k=1}^{m} C_k < \frac{a_0^2 - 2B}{2a_0}$.

Then zero is a global attractor for all solutions of problem (20), (17).

Theorem 6. *Consider the equation*

$$\ddot{x}(t) + f\big(t, x(t), \dot{x}(t)\big) + \sum_{k=1}^{m} s_k(t, x\big(h_k(t)\big), \dot{x}(t)) = 0, \tag{21}$$

where

$$f(t, v, 0) = 0, s_k(t, 0, u) = 0, 0 < a_0 \leq \frac{f(t,v,u)}{u} \leq A,$$

$$0 < b_k \leq \frac{s_k(t,v,u)}{v} \leq B_k, u \neq 0, t - h_k(t) \leq \tau.$$

Suppose at least one of the following conditions holds:

1) $\sum_{k=1}^{m} B_k \leq \frac{a_0^2}{4}, \frac{a_0}{2}(A - a_0) < \sum_{k=1}^{m} b_k - a_0 \sum_{k=1}^{m} B_k \tau_k$,
2) $\sum_{k=1}^{m} b_k > \frac{a}{2}\left(A - \frac{a_0}{2}\right), \sum_{k=1}^{m} B_k(1 + a_0\tau_k) < \frac{a_0^2}{2}$.

Then zero is a global attractor for all solutions of problem (21), (17).

Theorem 7. *Consider the equation*

$$\ddot{x}(t) + f\big(t, x(t), \dot{x}(t)\big) + s\big(t, x(t)\big) = \sum_{k=1}^{m} c_k(t)(x(t) - x(h_k(t))), \tag{22}$$

where

$$f(t, v, 0) = 0, s(t, 0) = 0, 0 < a_0 \leq \frac{f(t,v,u)}{u} \leq A,$$

$$0 < b_0 \leq \frac{s(t,u)}{u} \leq B, |c_k(t)| \leq C_k, u \neq 0, t - h_k(t) \leq \tau_k.$$

Suppose at least one of the following conditions holds:

1) $B \le \frac{a_0^2}{4}, \sum_{k=1}^{m} C_k\tau_k < \frac{2b_0 - a_0(A-a_0)}{2a_0}$,

2) $b_0 \ge \frac{a_0}{2}\left(A - \frac{a_0}{2}\right), \sum_{k=1}^{m} C_k\tau_k < \frac{a_0^2-2B}{2a_0}$.

Then zero is a global attractor for all solutions of problem (22), (17).

Example 1. *To illustrate Part 2) of Theorem 7, consider the equation*

$$\ddot{x}(t) + (1.9 + 0.1\sin x(t))\dot{x}(t) + (1.1 + 0.1\cos x(t))x(t - 0.19\sin^2 t) = 0. \quad (23)$$

We have $m = 1, a_0 = 1.8, A = 2, b_0 = 1, B = 1.2, \tau = 0.19$; *therefore, all conditions of the theorem hold, hence zero is a global attractor for all solutions of equation (23).*

Consider a generalized Kaldor-Kalecki model

$$\ddot{x}(t) + [\alpha(t) - \beta(t)p'(x(t))]\dot{x}(t) + s(t, x(t)) = p(x(t)) - p(x(h(t))), \quad (24)$$

where α, β are locally essentially bounded functions, s is a Caratheodory function, p is a locally absolutely continuous nondecreasing function,

$0 < \alpha_0 \le \alpha(t) \le \alpha_1, 0 < \beta_0 \le \beta(t) \le \beta_1$,

$|p'(t)| \le C, \alpha_0 - \beta_1 C > 0, 0 < b_0 \le \frac{s(t,u)}{u} \le B, t - h(t) \le \tau$.

Denote $a_0 = \alpha_0 - \beta_1 C$.

Theorem 8. *Suppose at least one of the following conditions holds:*

1) $B \le \frac{a_0^2}{4}, C_\tau < \frac{2b_0 - a_0(\alpha_1 - a_0)}{2a_0}$,

2) $b \ge \frac{a_0}{2}\left(\alpha_1 - \frac{a_0}{2}\right), C_\tau < \frac{a_0^2 - 2B}{2a_0}$.

Then zero is a global attractor for all solutions of problem (24), (17).

Proof. Suppose x is a fixed solution of problem (24),(17). There exists a function $\xi(t)$ such that $p(x(t)) - p(h(x(t))) = p'(\xi(t))\left(x(t) - x(h(t))\right)$. Denote $\alpha(t) - \beta(t)p'(x(t)) = a(t), p'(\xi(t)) = c(t)$.

Hence x is a solution of the following equation

$$\ddot{y}(t) + a(t)\dot{y}(t) + s(t, y(t)) = c(t)\left(y(t) - y(h(t))\right). \quad (25)$$

Since $p'(x) \ge 0$ then $0 < \alpha_0 - \beta_1 C \le a(t) \le \alpha_1$. Equation (25) has a form (22) with $f(t, x(t), \dot{x}(t)) = a(t)\dot{x}(t), m = 1$. All conditions of Theorem 7 hold, hence for any solution of (25) we have $\lim_{t\to\infty} y(t) = 0$. Then also $\lim_{t\to\infty} x(t) = 0$.

3. Remarks and Open Problems

The technique of reduction of a high-order linear differential equation to a system by the substitution $x^{(k)} = y_{k+1}$ is quite common. However, this substitution does not depend on the parameters of the original equation, and therefore does not offer new insight from a qualitative analysis point of view. Instead, we proposed a substitution which exploits the parameters of the original model. By using that approach, a broad class of the second order non-autonomous linear equations with delays was examined and explicit easily-verifiable sufficient stability conditions were obtained. There is a natural extension of this approach to stability analysis of high-order models. For the nonlinear second order non-autonomous equations with delays we applied the linearization technique and the results obtained for linear models. Our stability tests are applicable to some milling models and to a non-autonomous Kaldor–Kalecki business cycle model. Several numerical examples illustrate the application of the stability tests. We suggest that a similar technique can be developed for higher order linear delay equations, with or without non-delay terms.

Solution of the following problems will complement the results of the present paper:

1. In all stability conditions obtained, we used lower and upper bounds of the coefficients and the delays. It is interesting to obtain stability conditions in an integral form.

2. Apply the technique used in the paper to examine delay differential equations of higher order. Also, the substitution used in this chapter was based on the existence of a non-delay term, it would be interesting to adjust the method for equations which have several delayed terms only.

3. Establish necessary stability conditions for the equations considered in this chapter by reduction to a system of delay differential equations.

4. For the sunflower equation and its modifications establish set of conditions to guarantee boundedness of all solutions.

5. Apply the technique used in the paper to examine delay differential equations of higher order.

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ПРОСТЫЕ ТЕСТЫ УСТОЙЧИВОСТИ ДЛЯ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ЗАДЕРЖКИ ВТОРОГО ПОРЯДКА

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Для линейных и нелинейных дифференциальных уравнений запаздывания второго порядка с затухающими членами получены экспоненциальная устойчивость и условия глобальной асимптотической устойчивости. Результаты основаны на новых достаточных условиях устойчивости для систем линейных уравнений. Результаты иллюстрируются численными примерами, а также приводится перечень соответствующих проблем для будущего исследования.

Предложена подстановка, в которой используются параметры исходной модели. Используя этот подход, широкий класс неавтономных линейных уравнений второго порядка с задержками был исследован и получены явные легко проверяемые достаточные условия устойчивости. Приводится естественное продолжение этого подхода к анализу устойчивости моделей высокого порядка. Для нелинейных неавтономных уравнений второго порядка с задержками применен метод линеаризации и получены результаты для линейных моделей. Приведенные тесты стабильности применимы к некоторым моделям фрезерования и к неавтономной модели бизнес-цикла Калдора – Калецкого. Мы предлагаем, чтобы аналогичная методика была разработана для линейных уравнений с условием линейной задержки или без задержки.

Ключевые слова: дифференциальные уравнения запаздывания второго порядка, экспоненциальная устойчивость, редукция систем.

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