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ON ONE EQUATION OF INTERNAL WAVES

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The Cauchy–Dirichlet problem is considered for the equation of internal waves. This equation has various applications in hydrodynamics, for example, in the study of waves in the ocean. The article provides an analytical study of one equation of internal waves. This equation characterizes propagation of waves in a homogeneous incompressible stratified fluid. The equation of internal waves is reduced to an abstract semilinear Sobolev type equation of the second order. The study of the equation is carried out within the framework of the theory of polynomially bounded operator pencils. In this work, we construct propagators for the equation of internal waves. Also, we present two model examples, where the domain *D* is represented in the form of a cylinder and a parallelepiped. The result of the work is an analytical solution to the considered cases for the equation of internal waves.

Keywords: internal wave equation; polynomially bounded pencils of operators; Sobolev-type equation; propagators.

1. Introduction

Equation of internal waves in a homogeneous incompressible non-rotating fluid is described by the Poincare equation

$$\Delta u_{tt} + N^2\left(u_{xx} + u_{yy}\right) = 0, \tag{1}$$

where N^2 is buoyancy frequency. Earlier, the equation was considered in the works of S.A. Gabov and P.A. Krutitsky [1] when working with the excitation of nonstationary internal waves in a two-layer model of a stratified fluid, the behavior of the solution at a large time was studied. Yu.D. Pletner [2] studied some initial-boundary value problems and the fundamental solution to the equation of internal waves in the case of an incompressible exponentially stratified fluid. L.V. Perova and A.G. Sveshnikov [3] generalized the works of the authors devoted to the propagation of small perturbations in stratified and rotating fluids. The equation of internal waves can be written in another form, for example, given in [4]. Interest in the study of problems with the equation of internal waves is still topical. This problem is quite interesting from a mathematical point of view, since the solution to the Cauchy-Dirichlet problem is obtained within the framework of the theory of polynomially bounded operator pencils.

Let D be a bounded domain from R^3 with a smooth boundary ∂D. Consider the Dirichlet condition

$$u\left(x,y,z,t\right) = 0, \left(x,y,z,t\right) \in \partial D \times R \tag{2}$$

and the Cauchy condition

$$u\left(x,y,z,0\right) = u_0, u_t\left(x,y,z,0\right) = u_1. \tag{3}$$

The solution to problem (1)–(3) is found in the framework of the theory of Sobolev-type equations. In this paper, we use methods based on the theory of semigroups (groups) of operators [5, 6]. The monographs [6, 7] present the results of studying Sobolev-type equations and equations that are not solved with respect to the high-order derivative. In the paper [7] different classes of Sobolev-type equations are introduced and, by structure, equation (1) is considered to be a simple Sobolev-type equation.

2. (A, p)-bounded Operators

Let U,F be Banach spaces and the operators $A, B_0, B_1, \ldots, B_{n-1} \in L\left(F;U\right)$. Denote by $\vec{B}$ a pencil formed by the operators $B_0, B_1, ..., B_{n-1}$. The sets $\rho^A\left(\vec{B}\right) = \{\mu \in \mathbb{C} : \left(\mu^n A - \mu^{n-1}B_{n-1} - ... - \mu B_1 - B_0\right)^{-1} \in L\left(F;U\right)\}$ and $\sigma^A\left(\vec{B}\right) = \mathbb{C} \setminus \rho^A\left(\vec{B}\right)$ are called an A-resolvent set and an A-spectrum of the pencil $\vec{B}$, respectively. The operator-function of a complex variable $R_\mu^A\left(\vec{B}\right)=\left(\mu^n A-\mu^{n-1}B_{n-1}-...-\mu B_1-B_0\right)^{-1}$ with the domain $\rho^A\left(\vec{B}\right)$ is called an *A*-resolvent of the pencil $\vec{B}$.

Definition 1. [9] The operator pencil $\vec{B}$ is called polynomially bounded with respect to an operator A (or polynomially A-bounded) if $\exists a\in\mathbb{R}_+\ \forall\mu\in\mathbb{C}\left(|\mu|>a\right)\Rightarrow\left(R_\mu^A\left(\vec{B}\right)\in L\left(F;U\right)\right)$.

Remark 1. [9] If there exists an operator $A^{-1}\in L\left(F;U\right)$) then the pencil B is polynomially A-bounded.

To decompose the spaces U,F into a direct sum of subspaces, it is necessary to construct projectors. Condition (4) is necessary for the existence of projectors [8].

$$\int_\gamma \mu^k R_\mu^A\left(\vec{B}\right)\equiv\mathbb{O},\ k=0,1,\ldots,n-2, \tag{4}$$

where the circuit $\gamma=\{\mu\in C:|\mu|=r>a\}$.

Lemma 1. [9] Let the operator pencil $\vec{B}$ be polynomially *A*-bounded and condition (4) be fulfilled. Then the operators

$$P=\frac{1}{2\pi i}\int_\gamma R_\mu^A\left(\vec{B}\right)\mu^{n-1}Ad\mu, Q=\frac{1}{2\pi i}\int_\gamma \mu^{n-1}AR_\mu^A\left(\vec{B}\right)d\mu$$

are projectors in the spaces U and F respectively.

Denote $U^0=\ker P, F^0=\ker Q, U^1=\operatorname{im}P, F^1=\operatorname{im}Q$. According to Lemma 1, $U=U^0\oplus U^1, F=F^0\oplus F^1$. Denote by $A^k\left(B_l^k\right)$ a restriction of the operator $A\left(B_l\right)$ on $U^k\left(F^k\right), k=0,1; l=0,1,...,n-1.$

Theorem 1. [9] Let the operator pencil $\vec{B}$ be polynomially *A*-bounded and condition (4) be fulfilled. Then

(*i*) $A^k\in L\left(U^k,F^k\right), k=0,1;$

(*ii*) $B_l^k\in L\left(U^k,F^k\right), k=0,1, l=0,1,...,n-1;$

(*iii*) the operator $\left(A^1\right)^{-1}\in L\left(U^1,F^1\right)$ exists;

(iv) the operator $\left(B_0^0\right)^{-1}\in L\left(U^0,F^0\right)$ exists.

Using Theorem 1, construct the operators: $H_0=\left(B_0^0\right)^{-1}A^0\in L\left(U^0\right)$, $H_1=\left(B_0^0\right)^{-1}B_1^0\in L\left(U^0\right),...,H_{n-1}=\left(B_0^0\right)^{-1}B_{n-1}^0\in L\left(U^0\right)$ and $S_0=\left(A^1\right)^{-1}B_0^1\in L\left(U^1\right), S_1=\left(A^1\right)^{-1}B_1^1\in L\left(U^1\right),...,S_{n-1}=\left(A^1\right)^{-1}B_{n-1}^1\in L\left(U^1\right)$.

Definition 2. [9] Define the family of operators $\{K_q^1,K_q^2,...,K_q^n\}$ as follows: $K_0^s=\mathbb{O}, s\neq n, K_0^n=\mathbb{I}$,

$$K_1^1=H_0, K_1^2=-H_1,...,K_1^s=-H_{s-1},...,K_1^n=H_{n-1},$$

$$K_q^1=K_{q-1}^n H_0, K_q^2=K_{q-1}^1-K_{q-1}^n H_1,\ldots,K_q^s=K_{q-1}^{s-1}-K_{q-1}^n H_{s-1},\ldots,\ K_q^s=K_{q-1}^{n-1}-K_{q-1}^n H_{n-1}, q=1,2,...$$

The *A*-resolvent of the pencil $\vec{B}$ can be represented by the Laurent series

$$\left(\mu^n A-\mu^{n-1}B_{n-1}-...-\mu B_1-B_0\right)^{-1}=-\sum_{q=0}^{\infty}\mu^q K_q^n\left(B_0^0\right)^{-1}\left(\mathbb{I}-Q\right)+$$

$$+\sum_{q=1}^{\infty}\mu^{-q}\left(\mu^{n-1}S_{n-1}+...+\mu S_1+S_0\right)^q L_1^{-1}Q.$$

Definition 3. [9] The point ∞ is called

(i) a removable singularity of an A-resolvent of the pencil $\vec{B}$, if $K_1^s\equiv\mathbb{O}, s=1,2,...,n$;

(ii) a pole of the order $p\in N$ of an A-resolvent of the pencil $\vec{B}$, if $\exists p: K_p^s\not\equiv\mathbb{O}, s=1,2,...,n,$ but $K_{p+1}^s\equiv\mathbb{O}, s=1,2,...,n;$

(iii) an essential singularity of an A-resolvent of the pencil $\vec{B}$, if $K_q^n\not\equiv\mathbb{O}\ \forall q\in N$.

Further, for brevity, a removable singularity of an A-resolvent of the pencil $\vec{B}$ is called a pole of the order 0. If the operator pencil $\vec{B}$ is polynomially A-bounded and the point ∞ is a pole of the order $p\in\{0\}\cup N$ of an A-resolvent of the pencil $\vec{B}$, then the operator pencil $\vec{B}$ is called polynomially (A,p)-bounded.

3. Abstract Problem

Consider the Cauchy problem

$$u(0)=u_0, u_t(0)=u_1 \tag{5}$$

for the second-order Sobolev-type equation

$$Au_{tt}=B_1u_t+B_0u. \tag{6}$$

Operator-functions U_k^t, k = 0, 1, of the form

$$U_0^t=\frac{1}{2\pi i}\int_\gamma R_\mu^A(B)(\mu A-B_1)e^{\mu t}d\mu, \tag{7}$$

$$U_1^t=\frac{1}{2\pi i}\int_\gamma R_\mu^A(B)Ae^{\mu t}d\mu \tag{8}$$

are propagators. Here the circuit $\gamma\in\mathbb{C}$ bounds a domain containing the A-spectrum of the operator pencil $\vec{B}$. The solution to problem (5), (6) in terms of the theory of degenerate groups was obtained in [8], under the condition that the operator pencil $\vec{B}$ is polynomially bounded with respect to the operator A.

Theorem 2. [10] Let the operator pencil $\vec{B}$ be polynomially A-bounded, condition (4) be fulfilled, and ∞ be a pole of the order $p\in\{0\}\cup N$ of the A-resolvent of $\vec{B}$. There exists a unique solution $u\in C^\infty(\mathbb{R},U)$ to problem (5), (6) of the form

$$u(t)=U_1u_1+U_0u_0, \tag{9}$$

where $u_k\in\text{im}U_1^0=\text{im}U_0^0, k=0,1, \text{im}U_1^0$ and $\text{im}U_0^0$ is a subspace in U.

4. Internal Wave Equation

Consider the cases when the domain D is a parallelepiped or a cylinder. Let the domain $D=[0,a]\times[0,b]\times[0,c]\subset R^3$ be the parallelepiped. Problem (1)–(3) can be reduced to Cauchy problem (5) for equation (6).

Introduce the spaces $U=W_2^{l+2}(D), F=W_2^l(D)$ and define the operators in the given spaces

$$A=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}, B_1=\mathbb{O}, B_0=-N^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right).$$

For any $l\in\{0\}\cup N$, the operators $A,B_1,B_0\in L(U,F)$. Define $-\lambda_{k,m,n}^2=-(\pi k/a)^2-(\pi m/b)^2-(\pi n/c)^2$ to be the eigenvalues of the Dirichlet problem for the Laplace operator. Denote by $\varphi_{k,m,n}=\sin(\pi kx/a)\sin(\pi my/b)\sin(\pi nz/c)$ the orthogonal eigenfunctions that correspond to $\{-\lambda_{k,m,n}^2\}$ in the sense of the scalar product in $L^2(D)$.

Since $\{\varphi_{k,m,n}\} \subset C^{\infty}(D)$, then

$$\mu^2 A - \mu B_1 - B_0 = \sum_{k,m,n=1}^{\infty} \left[-\mu^2 \lambda_{k,m,n}^2 + N^2 \lambda_{k,m}^2 \right] < \varphi_{k,m,n}, \cdot > \varphi_{k,m,n},$$

where $B_0 \varphi_{k,m,n} = -\lambda_{k,m}^2 \varphi_{k,m,n}$, and $<\cdot,\cdot>$ is the inner product in $L^2(D)$. Construct the equation to determine the A-spectrum:

$$-\lambda_{k,m,n}^2 \mu^2 + N^2 \lambda_{k,m}^2 = 0, \;\; \mu_{k,m,n}^{1,2} = \pm \frac{N\lambda_{k,m}}{\sqrt{\lambda_{k,m,n}^2}} i.$$

The A-spectrum $\sigma^A(\vec{B}) = \{\mu_{k,m,n}^{1,2}\}$ is bounded, because $\left|\mu_{k,m,n}^{1,2}\right| \le N$. Since the operator A is continuously invertible in the given spaces, then condition (4) is satisfied. As a result, the conditions of Lemma 1 hold.

Construct propagators by formulas (7), (8) as follows:

$$U_0^t u_0 = \sum_{k,m,n=1}^{\infty} \cos\left(\frac{N\lambda_{k,m}}{\sqrt{\lambda_{k,m,n}^2}} t \right) < \varphi_{k,m,n}, u_0 > \varphi_{k,m,n},$$

$$U_1^t u_1 = \sum_{k,m,n=1}^{\infty} \frac{N\lambda_{k,m}}{\sqrt{\lambda_{k,m,n}^2}} \sin\left(\frac{N\lambda_{k,m}}{\sqrt{\lambda_{k,m,n}^2}} t \right) < \varphi_{k,m,n}, u_1 > \varphi_{k,m,n}.$$

By Theorem 2, the solution to problem (1)–(3) has the form

$$u(x,y,z,t) = \sum_{k,m,n=1}^{\infty} \cos\left(\frac{N\lambda_{k,m}}{\sqrt{\lambda_{k,m,n}^2}} t \right) < \varphi_{k,m,n}, u_0 > \varphi_{k,m,n} +$$

$$+ \sum_{k,m,n=1}^{\infty} \frac{N\lambda_{k,m}}{\sqrt{\lambda_{k,m,n}^2}} \sin\left(\frac{N\lambda_{k,m}}{\sqrt{\lambda_{k,m,n}^2}} t \right) < \varphi_{k,m,n}, u_1 > \varphi_{k,m,n}.$$

Now consider the case when the domain D is a cylinder. Similarly, as in the case of the parallelepiped, problem (1)–(3) can be reduced to Cauchy problem (5) for equation (6). Use the operators A, B_1, B_0 of the form

$$A = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}, B_1 = O, B_0 = -N^2 \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} \right).$$

Write down the equation to determine points of the A-spectrum:

$$\mu^2 \left((\nu_k^{(n)})^2 + (\pi m/l)^2 \right) + N^2 \left(\nu_k^{(n)} \right)^2 = 0,$$

where $B_0 \phi_{k,m,n} = \nu_k^{(n)} \phi_{k,m,n}$ and $\langle \cdot,\cdot \rangle$ is the scalar product in $L^2(D)$. We get the A-spectrum of the form

$$\mu_{k,m,n}^{1,2} = \pm \left(N\nu_k^{(n)} \Big/ \sqrt{(\nu_k^{(n)})^2 + (\pi m/l)^2} \right) i.$$

Let us construct propagators according to formulas (7), (8) as follows:

$$U_0^t u_0 = \sum_{k,m,n=1}^{\infty} \cos\left(\frac{N\nu_k^{(n)}}{\sqrt{(\nu_k^{(n)})^2 + (\pi m/l)^2}} t \right) < \varphi_{k,m,n}, u_0 > \varphi_{k,m,n},$$

$$U_1^t u_1 = \sum_{k,m,n=1}^{\infty} \frac{\sqrt{(\nu_k^{(n)})^2 + (\pi m/l)^2}}{N\nu_k^{(n)}} \sin\left(\frac{N\nu_k^{(n)}}{\sqrt{(\nu_k^{(n)})^2 + (\pi m/l)^2}} t \right) < \varphi_{k,m,n}, u_1 > \varphi_{k,m,n}.$$

By Theorem 2, the solution to problem (1)–(3) has the form

$$u(x,y,z,t)=\sum_{k,m,n=1}^{\infty}\cos\left(\frac{N\nu_k^{(n)}}{\sqrt{(\nu_k^{(n)})^2+(\pi m/l)^2}}t\right)<\varphi_{k,m,n},u_0>\varphi_{k,m,n}+$$

$$+\sum_{k,m,n=1}^{\infty}\frac{\sqrt{(\nu_k^{(n)})^2+(\pi m/l)^2}}{N\nu_k^{(n)}}\sin\left(\frac{N\nu_k^{(n)}}{\sqrt{(\nu_k^{(n)})^2+(\pi m/l)^2}}t\right)<\varphi_{k,m,n},u_1>\varphi_{k,m,n}.$$

As a result of the work, we obtain solutions to initial-boundary value problem (2)–(3) in a closed form on the considered domains for equation of internal waves (1).

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ОБ ОДНОМ УРАВНЕНИИ ВНУТРЕННИХ ВОЛН

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В статье приводится аналитическое исследование одного уравнения внутренних волн, в некоторых источниках именуемое уравнением Пуанкаре, выведенное из основной системы гидродинамики. Данное уравнение характеризует распространение волн в толще однородной несжимаемой стратифицированной и, в отличии от уравнения Соболева, невращающейся жидкости. Рассмотрен случай, когда частота плавучести есть величина постоянная. Для уравнения внутренних волн рассматривается задача Коши–Дирихле. Данное уравнение имеет различные приложения в гидродинамике, например, при исследовании волн в океане. Исследование уравнения проводится в рамках теории полиномиально ограниченных пучков операторов. Уравнение внутренних волн редуцируется к задаче Коши абстрактному полулинейному уравнению соболевского типа второго порядка. Затем показывается, что решение поставленной задачи удовлетворяет абстрактной теории. Далее рассмотрены два примера. В первом примере область ограничена параллелепипедом, а во втором – цилиндром. Для каждого случая области показано, что относительный спектр пучка операторов ограничен, частотой плавучести. После строятся пропагаторы, разрешающие оператор-функции, для уравнения внутренних волн для каждой из областей. Подставив начальные данные в пропагаторы, получим аналитическое решение задачи Коши для уравнения внутренних волн.

Ключевые слова: уравнение внутренних волн; полиномиально ограниченный пучок операторов; уравнение соболевского типа; пропагаторы.

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