Краткие сообщения

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A SHORT PROOF OF COMPLETION THEOREM FOR METRIC SPACES

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The completion theorem for metric spaces is always proven using the space of Cauchy sequences. In this paper, we give a short and alternative proof of this theorem via Zorn's lemma. First, we give a way of adding one point to an incomplete space to get a chosen non-convergent Cauchy sequence convergent. Later, we show that every metric space has a completion by constructing a partial ordered set of metric spaces.
Keywords: Completion theorem; metric space; complete space; Zorn's lemma.

Introduction

The completion theorem for metric spaces states that every metric space can be embedded in a complete metric space and the original space's image is dense in that complete space. In several sources, the proof is given by a classical method based on the space of the equivalence classes of all the Cauchy sequences, denoted by $\hat{X}$, of the given metric space X, see [1]. In this proof, it is shown that $\hat{X}$ is a complete metric space, X can be embedded in $\hat{X}$, the image of X is dense in $\hat{X}$ and this completion is unique up to isometry.

In this paper, we propose another way to prove the completion theorem by not using the embedding. We prove that if a metric space X is not complete, there exists a complete metric space $\hat{X}$ including X such that X is dense in $\hat{X}$, i.e. $\overline{X} = \hat{X}$.

The proof

Lemma 1. Let (X,d) be a metric space and (x_n) be a Cauchy sequence that is not convergent in X. We define a new space $\tilde{X} = X \cup \{c\}$ with the metric $\tilde{d}$ as follows:

$$\tilde{d}(x,y) = \begin{cases} d(x,y), & if \ x,y \in X, \\ \lim\limits_{n\to\infty} d(x_n,y), & if \ x = c, y \in X, \\ \lim\limits_{n\to\infty} d(x,x_n), & if \ x \in X, y = c, \\ 0, & if \ x = y = c, \end{cases}$$

where c is an element that does not belong to X. Then, $(\tilde{X},\tilde{d})$ is a metric space satisfying the properties $\tilde{d}(x,y) = d(x,y)$ for each $x,y \in X$, $\lim\limits_{n\to\infty} x_n = c$ and $\overline{X} = \tilde{X}$.

Proof. By the reverse triangle inequality $|d(x_n,y) - d(x_m,y)| \le d(x_n,x_m)$, the sequence $\{d(x_n,y)\}$ is a Cauchy sequence in $\mathbb{R}$ because (x_n) is a Cauchy sequence in X. Since $\mathbb{R}$ is complete, then $\{d(x_n,y)\}$ is a convergent sequence for each $y \in X$. Therefore, the function $\tilde{d}$ is well-defined from $\tilde{X} \times \tilde{X}$ to $\mathbb{R}$. Also, it is easy to see that $\tilde{d}$ is a metric on the set $\tilde{X}$. Now, we show that the sequence (x_n) converges to c in the metric space $(\tilde{X},\tilde{d})$. Given $\varepsilon > 0$, there exists a natural number n_ε such that $d(x_n,x_m) < \frac{\varepsilon}{2}$ for all $n,m > n_\varepsilon$. Then, we have

$$\tilde{d}(x_n, c) = \lim_{m\to\infty} d(x_n, x_m) \le \frac{\varepsilon}{2} < \varepsilon$$

for each $n > n_\varepsilon$. This proves the above assertion and also proves that $\bar{X} = \tilde{X}$.

Lemma 2. Let (X, d) be a metric space included two complete metric spaces (Y_1, ρ_1) and (Y_2, ρ_2) with the conditions $\rho_1(x, x^*) = d(x, x^*)$ and $\rho_2(x, x^*) = d(x, x^*)$ for every $x, x^* \in X$. If X is both dense in (Y_1, ρ_1) and (Y_2, ρ_2), then these complete metric spaces are isomorphic to each other.

Proof. Let $t \in Y_1$. Since $\overline{X} = Y_1$, there exists a sequence $(x_n) \subset X$ such that $\lim_{n\to\infty} \rho_1(x_n, t) = 0$. Then, (x_n) is a Cauchy sequence in Y_1, and also is a Cauchy sequence in Y_2. Since (Y_2, ρ_2) is a complete space, then there exists y in Y_2 such that $\lim_{n\to\infty} \rho_2(x_n, y) = 0$. Let $f : Y_1 \to Y_2$, $y = f(t)$. We now show that the function f is an isometry. First, we prove that it is well-defined. Let $(z_n) \subset X$ be also convergent to the point t in the space (Y_1, ρ_1). Then, similarly, there exists $z \in Y_2$ such that $\lim_{n\to\infty} \rho_2(z_n, z) = 0$.

$$\rho_2(y, z) = \rho_2\left(\lim_{n\to\infty} x_n, \lim_{n\to\infty} z_n\right) = \lim_{n\to\infty} \rho_2(x_n, z_n) = \lim_{n\to\infty} d(x_n, z_n) = \lim_{n\to\infty} \rho_1(x_n, z_n)$$

$$= \rho_1\left(\lim_{n\to\infty} x_n, \lim_{n\to\infty} z_n\right) = \rho_1(t, t) = 0 \Rightarrow y = z.$$

Second, we prove that f is surjective. Let $y \in Y_2$. Since $\overline{X} = Y_2$, there exists a sequence $(x_n) \subset X$ such that $\lim_{n\to\infty} \rho_2(x_n, y) = 0$. Then, (x_n) is a Cauchy sequence in X and also in Y_1. Since Y_1 is complete, then there exists $t \in Y_1$ such that $\lim_{n\to\infty} \rho_1(x_n, t) = 0$. By the construction of f, $y = f(t)$. Finally, we prove that f is an isometry. Let $t_1, t_2 \in Y_1$.Then, there exist two sequences $(x_n), (z_n) \subset X$ such that $\lim_{n\to\infty} \rho_1(x_n, t_1) = 0$, $\lim_{n\to\infty} \rho_1(z_n, t_2) = 0$, $\lim_{n\to\infty} \rho_2(x_n, f(t_1)) = 0$ and $\lim_{n\to\infty} \rho_2(z_n, f(t_2)) = 0$.

$$\rho_2(f(t_1), f(t_2)) = \rho_2\left(\lim_{n\to\infty} x_n, \lim_{n\to\infty} z_n\right) = \lim_{n\to\infty} \rho_2(x_n, z_n) = \lim_{n\to\infty} d(x_n, z_n) = \lim_{n\to\infty} \rho_1(x_n, z_n)$$

$$= \rho_1\left(\lim_{n\to\infty} x_n, \lim_{n\to\infty} z_n\right) = \rho_1(t_1, t_2).$$

This completes the proof.

Theorem 1. Every metric space has a unique completion up to isometry.

Proof. Consider the family Ω of metric spaces (Y, ρ) satisfying the following conditions:

1) ρ is a metric on Y,
2) $X \subset Y$,
3) $\rho(x, y) = d(x, y)$ for each $x, y \in X$,
4) $\bar{X} = Y$.

We define a relation on Ω which is as follows:

$$(Y_1, \rho_1) \le (Y_2, \rho_2) \Leftrightarrow Y_1 \subset Y_2 \text{ and } \rho_2(x, y) = \rho_1(x, y) \text{ for each } x, y \in Y_1.$$

It is easy to see that the pair $(\Omega, \le)$ is a poset. We take a chain Ω^* in the poset $(\Omega, \le)$ and define

$$Y^* = \bigcup_{(Y,\rho)\in\Omega^*} Y.$$

We now define a function ρ^* from $Y^* \times Y^*$ to $\mathbb{R}$ as follows. If $x, y \in Y^*$, then there exists $(Y_0, \rho_0) \in \Omega^*$ such that $x, y \in Y_0$ because Ω^* is a chain. Let

$$\rho^*(x, y) = \rho_0(x, y).$$

Then, the function ρ is well-defined by the definition of the relation $\leq$. One can easily show that ρ^* is a metric on the set Y^*. Let $y \in Y^*$, there exists $(\tilde{Y}, \tilde{\rho}) \in \Omega^*$ such that $y \in \tilde{Y}$. Since $(\tilde{Y}, \tilde{\rho}) \in \Omega$, then there exists a sequence (x_n) in X such that $\lim_{n\to\infty} \tilde{\rho}(x_n, y) = 0$, i.e. $\lim_{n\to\infty} \rho^*(x_n, y) = 0$. Consequently, $\bar{X} = Y^*$. These results show that the metric space (Y^*, ρ^*) belongs to Ω and forms an upper bound of the chain Ω^*. By Zorn's lemma, Ω has a maximal element and we denote it by $(\widehat{X}, \widehat{d})$. We now prove that the metric space $(\widehat{X}, \widehat{d})$ is a completion of (X, d). Since $(\widehat{X}, \widehat{d})$ is an element of Ω, then we just prove that $(\widehat{X}, \widehat{d})$ is complete. Assume the contrary. If $(\widehat{X}, \widehat{d})$ has a non-convergent Cauchy sequence, then Lemma 1 requires that there exists a metric space (X^*, d^*) such that $X^* = \widehat{X} \cup \{c^*\}$ and $(\widehat{X}, \widehat{d}) < (X^*, d^*)$, where c^* is a point not in $\widehat{X}$. We now show that $(X^*, d^*) \in \Omega$. Indeed, it is enough to show that c^* is an accumulation point of the original set X. Given $\varepsilon > 0$. By Lemma 1, there exists $c \in \widehat{X}$ such that $d^*(c, c^*) < \frac{\varepsilon}{2}$. Besides, by the relation $(\widehat{X}, \widehat{d}) \in \Omega$, there exists $x \in X$ such that $\widehat{d}(x, c) < \frac{\varepsilon}{2}$. Then, $d^*(x, c^*) \leq d^*(x, c) + d^*(c, c^*) = \widehat{d}(x, c) + d^*(c, c^*) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Thus, we have $(X^*, d^*) \in \Omega$. The last and the relation $(\widehat{X}, \widehat{d}) < (X^*, d^*)$ contradict the maximality of $(\widehat{X}, \widehat{d})$. This completes the proof.

The uniqueness up to isometry of the completion is directly obtained by Lemma 2.

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КОРОТКОЕ ДОКАЗАТЕЛЬСТВО ТЕОРЕМЫ О ПОПОЛНЕНИИ МЕТРИЧЕСКИХ ПРОСТРАНСТВ

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Приводится альтернативное доказательство теоремы о пополнении метрических пространств, основанное на лемме Цорна.

Ключевые слова: теорема о пополнении; метрическое пространство; полное пространство; лемма Цорна.

Литература

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