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EXACT SOLUTIONS OF THE HIROTA EQUATION USING THE SINE-COSINE METHOD

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Nonlinear partial differential equations of mathematical physics are considered to be major subjects in physics. The study of exact solutions for nonlinear partial differential equations plays an important role in many phenomena in physics. Many effective and viable methods for finding accurate solutions have been established.

In this work, the Hirota equation is examined. This equation is a nonlinear partial differential equation and is a combination of the nonlinear Schrödinger equation and the complex modified Korteweg–de Vries equation. The nonlinear Schrödinger equation is the physical model and occurs in various areas of physics, including nonlinear optics, plasma physics, superconductivity, and quantum mechanics. The complex modified Korteweg–de Vries equation has been applied as a model for the nonlinear evolution of plasma waves and represents the physical model that incorporates the propagation of transverse waves in a molecular chain model and in a generalized elastic solid.

To find exact solutions of the Hirota equation, the sine-cosine method is applied. This method is an effective tool for searching exact solutions of nonlinear partial differential equations in mathematical physics. The obtained solutions can be applied when explaining some of the practical problems of physics.

Keywords: Hirota equation; sine-cosine method; solution; ordinary differential equation; partial differential equation; nonlinearity.

Introduction

Nonlinear partial differential equations (PDEs) are widely used as models to describe physical phenomena in various fields of sciences such as biology, solid state physics, fluid mechanics, plasma physics, plasma wave, condensed matter physics, chemical physics, optical fibers, and chemical physics [1]. Various powerful methods such as, Darboux transformation method [2], Hirota's method [3] and sine-cosine method [1, 4–6], modification of the truncated expansion method [7], have been developed to obtain exact solutions of these equations.

In this work, we study the Hirota equation

$$iq_t + q_{xx} + 2|q|^2 q + i\alpha q_{xxx} + 6i\alpha |q|^2 q_x = 0, \quad (1)$$

where $q(x,t)$ is a complex valued function of the spatial coordinate x and the time t, α is a real constant, i is imaginary unit. The equation was introduced in [8] and studied in [9–11]. It is a combination of the nonlinear Schrödinger equation and the complex modified Korteveg–de Vrise equation. When $\alpha = 0$ the Hirota equation (1) can be reduced to the nonlinear Schrödinger equation.

1. Review of the sine-cosine method

In this section, we describe the sine-cosine method [1, 4–6]. According to the sine-cosine method by using a wave transformation

$$u(x,t) = u(\xi), \xi = (x - ct), \quad (2)$$

the partial differential equation (PDE)

$$E_1(u_t, u_x, u_{xx}, u_{xxx}, \ldots) = 0, \quad (3)$$

can be converted to ordinary differential equation (ODE)

$$E_2(u, u', u'', u''', \ldots) = 0 \quad (4)$$

Then the equation (4) is integrated as long as all terms contain derivatives where integration constants are considered zeros. The solutions of ODE (4) can be expressed in the form [1, 4–6]

$$u(x,t)=\begin{cases}\lambda\cos^{\beta}(\mu\xi), |\xi|\le \dfrac{\pi}{2\mu},\\ 0, \text{otherwise},\end{cases} \tag{5}$$

or

$$u(x,t)=\begin{cases}\lambda\sin^{\beta}(\mu\xi), |\xi|\le \dfrac{\pi}{2\mu},\\ 0, \text{otherwise},\end{cases} \tag{6}$$

where $\xi = x - ct$, the parameters μ and β will be determined, and μ is wave number and c is wave speed respectively [1]. The derivatives of (5) become

$$\left(u^n(\mu\xi)\right)' = -n\beta\mu\lambda^n\cos^{n\beta-1}(\mu\xi)\sin(\mu\xi), \tag{7}$$

$$\left(u^n(\mu\xi)\right)'' = -n^2\mu^2\beta^2\lambda^n\cos^{n\beta}(\mu\xi) + n\mu^2\lambda^n\beta(n\beta-1)\cos^{n\beta-2}(\mu\xi), \tag{8}$$

and the derivatives of (6) have next forms

$$\left(u^n(\mu\xi)\right)' = n\beta\mu\lambda^n\sin^{n\beta-1}(\mu\xi)\cos(\mu\xi), \tag{9}$$

$$\left(u^n(\mu\xi)\right)'' = -n^2\mu^2\beta^2\lambda^n\sin^{n\beta}(\mu\xi) + n\mu^2\lambda^n\beta(n\beta-1)\sin^{n\beta-2}(\mu\xi), \tag{10}$$

and so on for the other derivatives. Applying (5)–(10) into the reduced ordinary differential equation (4) we obtain a trigonometric equation of $\cos^r(\mu\xi)$ or $\sin^r(\mu\xi)$ terms. Then, we determine the parameters by first balancing the exponents of each pair of cosine or sine to determine β. Next, we collect all coefficients of the same power in $\cos^r(\mu\xi)$ or $\sin^r(\mu\xi)$, where these coefficients have to vanish. The system of algebraic equations among the unknown β and μ will be given and from that, we can determine coefficients.

2. Implementation of the sine-cosine method

We consider the Hirota equation (1). By transformation

$$q = e^{i(ax+dt)}u(x,t), \tag{11}$$

where a,d are real constants, the equation (1) can be converted to

$$-du + iu_t + 2iau_x + u_{xx} - a^2u + 2u^3 - 3a^2i\alpha u_x - 3a\alpha u_{xx} + a^3\alpha u + i\alpha u_{xxx} - 6a\alpha u^3 + 6i\alpha u^2u_x = 0. \tag{12}$$

By separating real and imaginary parts in the equation (12) we obtain the system of equations

$$-du + u_{xx} - a^2u + 2u^3 - 3a\alpha u_{xx} + a^3\alpha u - 6a\alpha u^3 = 0, \tag{13}$$

$$u_t + 2au_x - 3a^2\alpha u_x + \alpha u_{xxx} + 6\alpha u^2u_x = 0. \tag{14}$$

Substituting the wave transformation

$$u(x,t) = u(\xi) = u(x-ct), \tag{15}$$

into system of equations (13)–(14) we get the following two ordinary differential equations:

$$\left(a^3\alpha - a^2 - d\right)u + (1-3a\alpha)u'' + 2(1-3a\alpha)u^3 = 0, \tag{16}$$

$$\left(-c + 2a - 3a^2\alpha\right)u' + \alpha u''' + 2\alpha\left(u^3\right)' = 0. \tag{17}$$

Integrate equation (17) once, with respect to ξ, yields

$$\left(-c + 2a - 3a^2\alpha\right)u + \alpha u'' + 2\alpha u^3 = L, \tag{18}$$

where L is constant of integration.

As the same function $u(\xi)$ satisfies both equations (16) and (18), we obtain the following constraint condition:

$$\frac{a^3\alpha - a^2 - d}{-c + 2a - 3a^2\alpha} = \frac{(1-3a\alpha)}{\alpha}, \qquad L = 0. \tag{19}$$

By using condition (19), we have

$$c = 2a - 3a^2\alpha - \frac{\left(a^3\alpha^2 - d\alpha - a^2\alpha\right)}{(1-3a\alpha)}. \tag{20}$$

We can rewrite second order ordinary differential equation (16) as

$$u'' + 2u^3 + \left(\frac{a^3\alpha - a^2 - d}{1-3a\alpha}\right)u = 0. \tag{21}$$

In next subsection, we solve the equation (21) by the sine-cosine method.

2.1. The sine solution

According to method the solution of the (21) can be found by transformation

$$u(\mu\xi) = \lambda \sin^{\beta}(\mu\xi). \tag{22}$$

To find sine solution we use (22) and its derivative

$$u''(\mu\xi) = -\mu^2\beta^2\lambda\sin^{\beta}(\mu\xi) + \mu^2\lambda\beta(\beta-1)\sin^{\beta-2}(\mu\xi). \tag{23}$$

Substitute (22) and (23) into (21) we get

$$-\mu^2\beta^2\lambda\sin^{\beta}(\mu\xi) + \mu^2\lambda\beta(\beta-1)\sin^{\beta-2}(\mu\xi) + 2\lambda^3\sin^{3\beta}(\mu\xi) + \left(\frac{a^3\alpha - a^2 - d}{1-3a\alpha}\right)\lambda\sin^{\beta}(\mu\xi) = 0. \tag{24}$$

Using the balance method, by equating the exponents of $\sin^j, (24)$ we determine β:

$$\beta - 1 \neq 0,$$
$$\beta - 2 = 3\beta \qquad \Rightarrow \qquad \beta = -1. \tag{25}$$

Substitute (25) in (24) we obtain next equation

$$-\mu^2\lambda\sin^{-1}(\mu\xi) + 2\mu^2\lambda\sin^{-3}(\mu\xi) + 2\lambda^3\sin^{-3}(\mu\xi) + \left(\frac{a^3\alpha - a^2 - d}{1-3a\alpha}\right)\lambda\sin^{-1}(\mu\xi) = 0. \tag{26}$$

Equating the coefficients of each pair of the sine functions, we find the following system of algebraic equations:

$$\sin^{-1}(\mu\xi): \qquad -\mu^2\lambda + \left(\frac{a^3\alpha - a^2 - d}{1-3a\alpha}\right)\lambda = 0, \tag{27}$$

$$\sin^{-3}(\mu\xi): \qquad 2\lambda\mu^2 + 2\lambda^3 = 0. \tag{28}$$

From (27)–(28) we have

$$\mu = \pm\sqrt{\left(\frac{a^3\alpha - a^2 - d}{1-3a\alpha}\right)}, \; \alpha = \frac{a^2 + d - \lambda^2}{a^3 - 3a\lambda^2}, \tag{29}$$

where a, d, λ are real numbers.

Substituting (25), (29) into (22) and then obtained expression into (11) we have the sine solution of the Hirota equation

$$q_1(x,t) = \pm e^{i(ax+dt)}\lambda\sin^{-1}\left(\sqrt{\frac{a^3\alpha - a^2 - d}{1-3a\alpha}}(x-ct)\right), \tag{30}$$

where $c = 2a - 3a^2\alpha - \frac{\left(a^3\alpha^2 - d\alpha - a^2\alpha\right)}{(1-3a\alpha)}, \; \alpha = \frac{a^2 + d - \lambda^2}{a^3 - 3a\lambda^2}.$

2.2. The cosine solution

To find cosine solution we use

$$u(\mu\xi) = \lambda\cos^{\beta}(\mu\xi), \tag{31}$$

and its second order derivative

$$u''(\mu\xi) = -\mu^2\beta^2\lambda\cos^{\beta}(\mu\xi) + \mu^2\lambda\beta(\beta-1)\cos^{\beta-2}(\mu\xi). \quad (32)$$

Substitute (31) and (32) into (21) we get

$$-\mu^2\beta^2\lambda\cos^{\beta}(\mu\xi) + \mu^2\lambda\beta(\beta-1)\cos^{\beta-2}(\mu\xi) + 2\lambda^3\cos^{3\beta}(\mu\xi) + \left(\frac{a^3\alpha - a^2 - d}{1-3a\alpha}\right)\lambda\cos^{\beta}(\mu\xi) = 0. \quad (33)$$

Using the balance method, by equating the exponents of cos^j, (33) we find β:

$$\beta - 1 \neq 0,$$
$$\beta - 2 = 3\beta \quad \Rightarrow \quad \beta = -1. \quad (34)$$

Substituting (34) in (33) we obtain next equation

$$-\mu^2\lambda\cos^{-1}(\mu\xi) + 2\mu^2\lambda\cos^{-3}(\mu\xi) + 2\lambda^3\cos^{-3}(\mu\xi) + \left(\frac{a^3\alpha - a^2 - d}{1-3a\alpha}\right)\lambda\cos^{-1}(\mu\xi) = 0. \quad (35)$$

From (35) equating the coefficients of each pair of the cosine functions, we find the following system of algebraic equations:

$$\cos^{-1}(\mu\xi): \quad -\mu^2\lambda + \left(\frac{a^3\alpha - a^2 - d}{1-3a\alpha}\right)\lambda = 0, \quad (36)$$

$$\cos^{-3}(\mu\xi): \quad 2\lambda\mu^2 + 2\lambda^3 = 0. \quad (37)$$

Solving system (36)–(37) leads to the results,

$$\mu = \pm\sqrt{\left(\frac{a^3\alpha - a^2 - d}{1-3a\alpha}\right)}, \quad \alpha = \frac{a^2 + d - \lambda^2}{a^3 - 3a\lambda^2}, \quad (38)$$

where a, d, λ are real numbers.

Substituting (34), (38) into (31) and then obtained expression into (11) we have the cosine solution

$$q_2(x,t) = \pm e^{i(ax+dt)}\lambda\cos^{-1}\left(\sqrt{\frac{a^3\alpha - a^2 - d}{1-3a\alpha}}(x-ct)\right), \quad (39)$$

where $c = 2a - 3a^2\alpha - \frac{(a^3\alpha^2 - d\alpha - a^2\alpha)}{(1-3a\alpha)}$, $\alpha = \frac{a^2 + d - \lambda^2}{a^3 - 3a\lambda^2}$.

Conclusion

The sine-cosine method was effectively used for the analytic treatment of the Hirota equation. Exact solutions were derived. The obtained solutions can have an application to some practical physical problems. As the Hirota equation is a combination of the nonlinear Schrödinger equation and the complex modified Korteveg-de Vries equation in case α = 0 we can get exact solutions for the nonlinear Schrödinger equation. The applied method can be used in further works to establish more entirely new solutions for other kinds of nonlinear evolution partial differential equations.

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ТОЧНЫЕ РЕШЕНИЯ УРАВНЕНИЯ ХИРОТА С ПОМОЩЬЮ МЕТОДА СИНУС–КОСИНУС

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Нелинейные дифференциальные уравнения в частных производных математической физики являются важным объектом в физике. Так, изучение точных решений нелинейных уравнений в частных производных играет важную роль во многих явлениях в физике. Существует множество эффективных и действенных методов нахождения точных решений.

В данной работе исследовано уравнение Хироты. Это уравнение является нелинейным уравнением в частных производных и представляет собой комбинацию нелинейного уравнения Шредингера и комплексного модифицированного уравнения Кортевега–де Фриза. Нелинейное уравнение Шредингера является физической моделью и встречается в различных областях физики, включая нелинейную оптику, физику плазмы, сверхпроводимость и квантовую механику. Комплексное модифицированное уравнение Кортевега–де Фриза применяется в качестве модели нелинейной эволюции плазменных волн и представляет собой физическую модель, которая включает распространение поперечных волн в модели молекулярной цепочки и в обобщенном упругом твердом теле.

Для нахождения точных решений уравнения Хироты применен метод синус-косинус. Этот метод является эффективным инструментом для поиска точных решений нелинейных уравнений в частных производных математической физики. Полученные решения могут иметь приложение для объяснения некоторых практических задач физики.

Ключевые слова: уравнение Хироты, метод синус-косинус, решение, обыкновенное дифференциальное уравнение, дифференциальное уравнение в частных производных, нелинейность.

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