Математика

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ON A q-BOUNDARY VALUE PROBLEM WITH DISCONTINUITY CONDITIONS

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In this paper, we studied *q*-analogue of Sturm–Liouville boundary value problem on a finite interval having a discontinuity in an interior point. We proved that the *q*-Sturm–Liouville problem is self-adjoint in a modified Hilbert space. We investigated spectral properties of the eigenvalues and the eigenfunctions of *q*-Sturm–Liouville boundary value problem. We shown that eigenfunctions of *q*-Sturm–Liouville boundary value problem are in the form of a complete system. Finally, we proved a sampling theorem for integral transforms whose kernels are basic functions and the integral is of Jackson's type.

Keywords: q-Sturm–Liouville operator; self-adjoint operator; completeness of eigenfunctions; sampling theory.

Introduction

Boundary value problems with discontinuity conditions on the interval often appear in mathematics and other branches of sciences. Quantum calculus was initiated at the beginning of the 19th century and in recent years, many papers subject to the boundary value problems consisting a q-Jackson derivative in the classical Sturm–Lioville problem have occured [1]. In [2, 3], q-Sturm–Liouville problems are investigated and a space of boundary values of the minimal operator and describe all maximal dissipative, self-adjoint, maximal accretive and other extensions of q-Sturm–Liouville operators in terms of boundary conditions are raised. A theorem on completeness of the system of eigenfunctions and associated functions of dissipative operators are proved by using the Lidskii's theorem.

Also, there are a lot of physical models involving q-difference and their related problems in [4, 5]. In [6], the construction of expansions in q-Fourier series was followed by the derivation of the q-sampling theorems. In [7], a q-version of the sampling theorem was derived using the q-Hankel transform. The sampling theory associated with q-type of Sturm–Liouville equations is conceived (see [8, 9]).

In [10], it is proved that the regular symmetric q-Sturm–Liouville operator is semi-bounded and investigated the continuous spectrum of this operator. In [11], authors established a Parseval equality and an expansion formula in eigenfunctions for a singular q-Sturm–Liouville operator.

In this paper, q-analogue of Sturm–Liouville boundary value problems with discontinuity conditions in an interior point ([12]) are discussed.

Let us consider the boundary value problem **L** for the equation:

$$\mathbf{l}(y) := -\frac{1}{q} D_{q^{-1}} D_q y(x) + v(x) y(x) = \nu y(x), \tag{1}$$

on the interval $x \in (0,T)$ with the boundary conditions

$$U(y) := D_{q^{-1}} y(0) - \gamma y(0) = 0, \quad V(y) := D_{q^{-1}} y(T) + \Gamma y(T) = 0, \tag{2}$$

together with the jump conditions at a point $a \in (0,T)$

$$y(a+0) = \alpha_1 y(a-0), \quad D_{q^{-1}} y(a+0) = \alpha_1^{-1} D_{q^{-1}} y(a-0) + \alpha_2 y(a-0). \tag{3}$$

Here $v(x) \in L_q^2(0,T)$ is a real-valued function, α_1, α_2, γ and Γ are real numbers; $\alpha_1 > 0$.

1. Preliminaries on q-calculus

In this section, we give some of the q-notations and we will use these q-notations throughout the paper. These standard notations are founded in [13].

Let q be a positive number with $0<q<1$. Let h be a real or complex valued function on A (A is q-geometric set (see [2])). The q-difference operator D_q (the Jackson q-derivative) is defined as

$$D_q h(x)=\frac{h(x)-h(qx)}{x(1-q)},\quad x\neq 0.$$

When required we will replace q by q^{-1}. We can demonstrate the correctness of the following facts using the definition and will use often

$$D_{q^{-1}}h(x)=(D_q h)(q^{-1}x),\quad D_q^2 h(q^{-1}x)=qD_q[D_q h(q^{-1}x)]=D_{q^{-1}}D_q h(x).$$

Let h and g be defined on a q-geometric set A such that the q-derivatives of h and g exist for all $x\in A$. Then, there is a non-symmetric formula for the q-differentiation of a product

$$D_q[h(x)g(x)]=h(qx)D_q g(x)+g(x)D_q h(x). \tag{4}$$

The q-integral usually associated with the name of Jackson is defined in the interval $(0,T)$, as

$$\int_0^T h(x)d_q x=(1-q)\sum_{n=0}^{\infty}h(Tq^n)Tq^n.$$

Let $L_q^2(0,T)$ be the space of all complex-valued functions defined on $(0,T)$, such that

$$\|h\|=\left(\int_0^T |h(x)|^2\, d_q x\right)^{\frac{1}{2}}<\infty.$$

The space $L_q^2(0,T)$ is a separable Hilbert space (see [6]) with the inner product

$$\langle h,g\rangle=\int_0^T h(x)\overline{g(x)}d_q x.$$

If h and g are both q-regular at zero, there is a rule of q-integration by parts given by

$$\int_0^T g(x)D_q h(x)d_q x=(hg)(T)-(hg)(0)-\int_0^T D_q g(x)h(qx)d_q x. \tag{5}$$

The q appearing in the argument of h in the right-hand side integrand is another manifestation of the symmetry that is everywhere present in q-calculus. As an important special case, we have

$$\int_0^T D_q h(x)d_q x=(h)(T)-(h)(0). \tag{6}$$

Lemma 1. (see [2]) Let $h(.)$, $g(.)$ in $L_q^2(0,T)$ be defined on $[0,q^{-1}T]$. Then, for $x\in(0,T]$ we have

$$(D_q h)(xq^{-1})=D_{q^{-1}}h(x), \tag{7}$$

$$\langle D_q h,g\rangle=h(T)\overline{g(Tq^{-1})}-\lim_{n\to\infty}h(Tq^n)\overline{g(Tq^{n-1})}+\langle h,-\frac{1}{q}D_{q^{-1}}g\rangle, \tag{8}$$

$$\langle -\frac{1}{q}D_{q^{-1}}h,g\rangle=\lim_{n\to\infty}h(Tq^{n-1})\overline{g(Tq^n)}-h(Tq^{-1})\overline{g(T)}+\langle h,D_q g\rangle. \tag{9}$$

2. Properties of the spectral characteristics

Let $h(x)$ and $g(x)$ be continuously differentiable functions on $[0,a]$ and $[a,T]$. Denote

$$W_q(h,g)(x)=\langle h,g\rangle:=h(x)D_q g(x)-g(x)D_q h(x).$$

Here $W_q(h,g)$ is defined as the q-Wronskian of two function h and g. If $h(x)$ and $g(x)$ satisfy the jump conditions (3), then

$$\langle h,g\rangle_{|x=a+0} = \langle h,g\rangle_{|x=a-0}, \tag{10}$$

i. e. the function $\langle h,g\rangle$ is continuous on $[0,T]$. Applying formula (4), we obtain

$$D_qW_q(h,g)(x) = D_q\left(h(x)D_qg(x) - g(x)D_qh(x)\right) = h(qx)D_q^2g(x) - g(qx)D_q^2h(x). \tag{11}$$

On the other hand,

$$\begin{aligned} D_qW_q(h,g)(q^{-1}x) &= h(x)D_q^2g(q^{-1}x) - g(x)D_q^2h(q^{-1}x) \\ &= qh(x)[v(x)g(x) - \nu g(x)] - qg(x)[v(x)h(x) - \nu h(x)] = 0. \end{aligned} \tag{12}$$

As a result,

$$0 = D_qW_q(h,g)(q^{-1}x) = \frac{W_q(h,g)(q^{-1}x) - W_q(h,g)(x)}{q^{-1}x(1-q)}, \tag{13}$$

so, for $x \neq 0$,

$$W_q(h,g)(x) = W_q(h,g)(q^{-1}x), \tag{14}$$

i. e. the q-Wronskian $W_q(h,g)(x)$ does not depend on x.

Let $\eta(x,\nu)$ and $\xi(x,\nu)$ be the solution of equation (1) under the boundary conditions

$$\eta(0,\nu) = \xi(T,\nu) = 1, \quad D_{q^{-1}}\eta(0,\nu) = \gamma, \quad D_{q^{-1}}\xi(T,\nu) = -\Gamma. \tag{15}$$

and under the jump conditions (3). Then

$$U(\eta) = V(\xi) = 0. \tag{16}$$

Since the q-Wronskian is independent of x, we can evaluate

$$W_q(\eta,\xi)(\nu) := W_q(\nu) = -V(\eta) = U(\xi). \tag{17}$$

$W_q(\nu)$ is called the characteristic function of **L**.

Lemma 2. The eigenvalues $\{\nu_n\}_{n\geq 0}$ of the boundary value problem **L** coincide with zeros of the characteristic function. The functions $\eta(x,\nu_n)$ and $\xi(x,\nu_n)$ are eigenfunctions and

$$\xi(x,\nu_n) = \beta_n\eta(x,\nu_n), \quad \beta_n \neq 0. \tag{18}$$

Denote

$$\alpha_n = \int_0^T \eta^2(x,\nu_n)d_qx. \tag{19}$$

The set $\{\nu_n,\alpha_n\}_{n\geq 0}$ is called the spectral date of **L**.

Lemma 3. The following relation holds

$$\beta_n\alpha_n = \dot{W}_q(\nu_n), \tag{20}$$

where $\dot{W}_q(\nu) = D_qW_q(\nu)$ (respect to ν).

The proof of Lemma 2 and Lemma 3 can be done similar to [12].

Theorem 1. The q-Sturm–Liouville eigenvalue problem (1)–(3) is self-adjoint on $C_q^2(0) \cap L_q^2(0,T)$.

Proof. We first prove that $h(.)$, $g(.)$ in $L_q^2(0,T)$, we have the following q-Lagrange's identity

$$\int_0^T \left(\mathbf{l}(h(x))\overline{g(x)} - h(x)\overline{\mathbf{l}(g(x))}\right)d_qx = [h,g](T) - \lim_{n\to\infty}[h,g](Tq^n), \tag{21}$$

where

$$[h,g](x) := h(x)\overline{D_{q^{-1}}g(x)} - D_{q^{-1}}h(x)\overline{g(x)}. \tag{22}$$

Applying (9) with $h(x) = D_qh(x)$ and $g(x) = g(x)$, we obtain

$$\begin{aligned} &< -\frac{1}{q}D_{q^{-1}}D_qh(x), g(x) > \\ &= \lim_{n\to\infty}(D_qh)(Tq^{n-1})\overline{g(Tq^n)} - (D_qh)(Tq^{-1})\overline{g(T)} + < D_qh, D_qg > \end{aligned}$$

$$= \lim_{n\to\infty} D_{q^{-1}} h(Tq^n)\overline{g(Tq^n)} - D_{q^{-1}} h(T)\overline{g(T)} + < D_q h, D_q g > . \quad (23)$$

Applying (8) with $h(x) = h(x)$, $g(x) = D_q g(x)$, we obtain

$$< D_q h, D_q g >= h(T)\overline{D_q g(Tq^{-1})} - \lim_{n\to\infty} h(Tq^n)\overline{D_q g(Tq^{n-1})} + < h, -\frac{1}{q} D_{q^{-1}} D_q g >$$
$$= h(T)\overline{D_{q^{-1}} g(T)} - \lim_{n\to\infty} h(Tq^n)\overline{D_{q^{-1}} g(Tq^n)} + < h, -\frac{1}{q} D_{q^{-1}} D_q g > . \quad (24)$$

Therefore,

$$< -\frac{1}{q} D_{q^{-1}} D_q h(x), g(x) >= [h,g](T) - \lim_{n\to\infty} [h,g](Tq^n) + < h, -\frac{1}{q} D_{q^{-1}} D_q g > . \quad (25)$$

Lagrange's identity (21) results from (25) and the reality of $v(x)$. Letting $h(.), g(.)$ in $C_q^2(0)$ and assuming the that they satisfy (2)–(3), we obtain

$$D_{q^{-1}} h(0) - \gamma h(0) = 0, \quad D_{q^{-1}} g(0) - \gamma g(0) = 0. \quad (26)$$

The continuity of $h(.), g(.)$ at zero implies that $\lim_{n\to\infty} [h,g](Tq^n) = [h,g](0)$. Then (25) will be

$$< -\frac{1}{q} D_{q^{-1}} D_q h, g >= [h,g](T) - [h,g](0) + < h, -\frac{1}{q} D_{q^{-1}} D_q g > .$$

From (26), we have

$$[h,g](0) = h(0)\overline{D_{q^{-1}} g(0)} - D_{q^{-1}} h(0)\overline{g(0)} = 0.$$

Similarly, from (2) we obtain

$$[h,g](T) = h(T)\overline{D_{q^{-1}} g(T)} - D_{q^{-1}} h(T)\overline{g(T)} = 0.$$

Since $v(x)$ is real-valued function, then

$$< \mathrm{l}(h), g > = < -\frac{1}{q} D_{q^{-1}} D_q h(x) + v(x)h(x), g(x) > = < -\frac{1}{q} D_{q^{-1}} D_q h(x), g(x) > + < v(x)h(x), g(x) >$$
$$= < h, -\frac{1}{q} D_{q^{-1}} D_q g > + < h(x), v(x)g(x) > = < h, \mathrm{l}(g) >,$$

i.e. $\mathbf{l}$ is a self-adjoint operator.

Lemma 4. The eigenvalues $\{\nu_n\}$ of the boundary value problem (1)–(3) are real. Eigenfunctions related to different eigenvalues are orthogonal in $L_q^2(0,T)$. All zeros of $W_q(\nu)$ are simple, i. e. $\dot{W}_q(\nu_n) \neq 0$.

Proof. Let ν_0 be an eigenvalue with an eigenfunction $\eta_0(.)$. Then,

$$< \mathbf{l}(\eta_0), \eta_0 >=< \eta_0, \mathbf{l}(\eta_0) > . \quad (27)$$

Since $\mathbf{l}(\eta_0) = \nu_0 \eta_0$, then

$$(\nu_0 - \overline{\nu_0}) \int_0^T |\eta_0(x)|^2 \, d_q x. \quad (28)$$

Since $\eta_0(.)$ is non-trivial then $\nu_0 = \overline{\nu_0}$. So the eigenvalues are real.

Let ν, μ be two distinct eigenvalues with corresponding eigenfunctions $\eta(.)$, $\xi(.)$, respectively. Then,

$$(\nu - \mu) \int_0^T \eta(x)\overline{\xi(x)} d_q x = 0.$$

Since $\nu \equiv \mu$, then $\eta(.)$ and $\xi(.)$ are orthogonal.

Since $\eta(x, \nu_n)$ and $\xi(x, \nu)$ are solutions of the boundary value problem (1)–(3), we obtain

$$D_q \langle \eta(x, \nu_n), \xi(x, \nu) \rangle = (\nu_n - \nu) \eta(x, \nu_n) \xi(x, \nu). \quad (29)$$

Integrating equation (29) from 0 to T and using the conditions (2), we obtain

$$\int_0^T \eta(x,\nu_n),\xi(x,\nu)d_qx = \frac{W_q(\nu_n)-W_q(\nu)}{\nu_n-\nu}.$$

Since $\xi(x,\nu_n)=\beta_n\eta(x,\nu_n)$ as $\nu\to\nu_n$, we obtain

$$\dot{W}_q(\nu_n)=\beta_n\alpha_n.$$

Thus it follows that $\dot{W}_q(\nu_n)\neq 0$.

3. Completeness of Eigenfunctions

Theorem 2. The system of eigenfunctions $\{\eta(x,\nu_n)\}_{n\geq 0}$ of the boundary value problem (1)–(3) is complete in $L_q^2(0,T)$.

Proof. Consider the function

$$Y(x,\nu)= \frac{1}{W_q(\nu)}\left[\xi(x,\nu)\int_0^x \eta(t,\nu)h(t)d_qt+\eta(x,\nu)\int_x^T \xi(t,\nu)h(t)d_qt\right].$$

It is easy to verify that

$$-\frac{1}{q}D_{q^{-1}}D_qY(x)+\{-\nu+v(x)\}Y(x)=h(x),\quad x\in[0,T],\quad \nu\in \mathrm{C}, \tag{30}$$

Furthermore, taking into account (19), from (18) and (20) we get

$$\begin{aligned} Res_{\nu=\nu_n}Y(x,\nu) &= \frac{1}{\dot{W}_q(\nu_n)}\left[\xi(x,\nu_n)\int_0^x \eta(t,\nu_n)h(t)d_qt+\eta(x,\nu_n)\int_x^T \xi(t,\nu_n)h(t)d_qt\right] \\ &= \frac{\beta_n}{\dot{W}_q(\nu)}\eta(x,\nu_n)\int_0^T \eta(t,\nu_n)h(t)d_qt = \frac{1}{\alpha_n}\eta(x,\nu_n)\int_0^T \eta(t,\nu_n)h(t)d_qt. \end{aligned} \tag{31}$$

Let the function $h(x)\in L_q^2(0,T)$ be such that

$$\int_0^T \eta(t,\nu_n)h(t)d_qt=0,\quad n=0,1,2,... \tag{32}$$

Then in view of (31), $Res_{\nu=\nu_n}Y(x,\nu)=0$ and consequently for each fixed $x\in[0,T]$, the function $Y(x,\nu)$ is entire in ν. Furthermore, for $\rho\in G_\delta=\{\rho:|\rho-\rho_n|\geq\delta\}$ and $|\rho|\geq\rho^*$ where $\nu=\rho^2$, ρ_n are the zeros of the function

$$W_q^0(\rho)=\rho\big(b_1\sin\rho T-b_2\sin\rho(2a-T)\big),$$

where $b_1=\dfrac{\alpha_1+\alpha_1^{-1}}{2}$, $b_2=\dfrac{\alpha_1-\alpha_1^{-1}}{2}$, δ is a fixed positive number, ρ^* is rather large, the inequality

$$|W_q(\nu)|\geq C_\delta|\rho|e^{|\tau|T},\quad \rho\in C_\delta,\quad \rho=\sigma+i\tau$$

and consequently the inequality

$$|Y(x,\nu)|\leq\frac{C_\delta}{|\rho|},\quad \rho\in G_\delta,$$

is obtained (see [12]). Using the maximum principle and Liouville's theorem we conclude that $Y(x,\nu)\equiv 0$. From this and (30) it follows that $h(x)=0$ a. e. on $(0,T)$. Thus the theorem is proved.

4. The q-sampling theory

Theorem 3. Let $\eta(x,\nu)$ and $\xi(x,\nu)$ be the solutions of (1) selected as above. Then every function h of the form

$$h(\nu)=\int_0^T v(x)\eta(x,\nu)d_qx,\quad v\in L_q^2(0,T), \tag{33}$$

can be written as the Lagrange-type sampling expansion

$$h(\nu)=\sum_{n=0}^{\infty}h(\nu_n)\frac{W_q(\nu)}{\dot{W}_q(\nu_n)(\nu-\nu_n)},\tag{34}$$

where $W_q(\nu)$ is the q-Wronskian of the functions $\eta(x,\nu)$ and $\xi(x,\nu)$.

Proof. We multiply equation (1) with $\eta(x,\nu_n)$. Then we consider again equation (1), but replace ν by ν_n and multiply this last equation by $\eta(x,\nu)$. Subtracting the two results yields

$$(\nu-\nu_n)\eta(x,\nu)\eta(x,\nu_n)=D_q^2\eta(q^{-1}x,\nu_n)\eta(x,\nu)-D_q^2\eta(q^{-1}x,\nu)\eta(x,\nu_n).$$

From the rule for the q-differentiation of product (4), we can write

$$(\nu-\nu_n)\eta(x,\nu)\eta(x,\nu_n)=D_q\left[D_q\eta(q^{-1}x,\nu_n)\eta(x,\nu)-D_q\eta(q^{-1}x,\nu)\eta(x,\nu_n)\right]$$

If we apply a q-integration by means of (6) we obtain

$$(\nu-\nu_n)\int_0^T\eta(x,\nu)\eta(x,\nu_n)d_qx=\int_0^T D_q\left[D_q\eta(q^{-1}x,\nu_n)\eta(x,\nu)-D_q\eta(q^{-1}x,\nu)\eta(x,\nu_n)\right]d_qx$$

$$=D_q\eta(q^{-1}T,\nu_n)\eta(T,\nu)-D_q\eta(q^{-1}T,\nu)\eta(T,\nu_n)-\left(D_q\eta(q^{-1}0,\nu_n)\eta(0,\nu)-D_q\eta(q^{-1}0,\nu)\eta(0,\nu_n)\right)$$

From the condition (2), we have

$$D_q\eta(q^{-1}0,\nu_n)\eta(0,\nu)-D_q\eta(q^{-1}0,\nu)\eta(0,\nu_n)=D_{q^{-1}}\eta(0,\nu_n)\eta(0,\nu)-D_{q^{-1}}\eta(0,\nu)\eta(0,\nu_n)$$

$$=D_{q^{-1}}\eta(0,\nu_n)-\gamma\eta(0,\nu_n)=U(\eta)=0.$$

Multiply (17) by $\eta(T,\nu_n)$ to obtain

$$W_q(\nu)\eta(T,\nu_n)\quad=-D_{q^{-1}}\eta(T,\nu)\eta(T,\nu_n)-\Gamma\eta(T,\nu)\eta(T,\nu_n)$$

$$=-D_{q^{-1}}\eta(T,\nu)\eta(T,\nu_n)+D_{q^{-1}}\eta(T,\nu_n)\eta(T,\nu).$$

Then, we get

$$(\nu-\nu_n)\int_0^T\eta(x,\nu)\eta(x,\nu_n)d_qx=\quad W_q(\nu)\eta(T,\nu_n).$$

as a result,

$$\int_0^T\eta(x,\nu)\eta(x,\nu_n)d_qx=\frac{W_q(\nu)\eta(T,\nu_n)}{\nu-\nu_n}$$

and taking the limit as $\nu\to\nu_n$ gives

$$\int_0^T|\eta(x,\nu_n)|^2\,d_qx=\dot{W}_q(\nu_n)\eta(T,\nu_n).$$

We can therefore apply Kramer's lemma (see [14]) and write an integral transform of the form (33) as

$$h(\nu)=\sum_{n=0}^{\infty}h(\nu_n)\frac{W_q(\nu)}{\dot{W}_q(\nu_n)(\nu-\nu_n)}.\tag{35}$$

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О q-ГРАНИЧНОЙ ЗАДАЧЕ С РАЗРЫВНЫМИ УСЛОВИЯМИ

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Изучается *q*-аналог граничной задачи Штурма–Лиувилля на конечном интервале, имеющем разрыв во внутренней точке. Доказывается, что *q*-граничная задача Штурма–Лиувилля является само-сопряженной в модифицированном Гильбертовом пространстве. Исследуются спектральные свойства собственных значений и собственных функций *q*-граничной задача Штурма–Лиувилля. Показано, что собственные функции *q*-граничной задача Штурма–Лиувилля представимы в виде полной системы. Наконец, доказывается теорема о дискретном представлении для интегральных преобразований, чьи ядра являются базисными функциями, а интеграл имеет тип Джексона.

Ключевые слова: q-оператор Штурма–Лиувилля; самосопряженный оператор; полнота собственных функций; теорема о дискретном представлении.

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