MSC 65N85 **DOI: 10.14529/mmph220107**

ANALYSIS OF THE BOUNDARY VALUE PROBLEM FOR THE POISSON EQUATION

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Abstract. The mixed boundary value problem for the Poisson equation is considered in a bounded flat domain. The continuation of this problem through the boundary with the Dirichlet condition to a rectangular domain is carried out. Consideration of the continued problem in the operator form is proposed. To solve the continued problem, a method of iterative extensions is formulated in an operator form. The extended problem in operator form is considered on a finite-dimensional subspace. To solve the previous problem, an iterative extension method is formulated in operator form on a finite-dimensional subspace. The continued problem is presented in matrix form. To solve the continued problem in matrix form, the method of iterative extensions in matrix form is formulated. It is shown that in the proposed versions of the method of iterative extensions, the relative errors converge in a rate that is stronger than the energy norm of the extended problem with the rate of geometric progression. The iterative parameters in these methods are selected using the minimum residual method. Conditions are indicated that are sufficient for the convergence of the applied iterative processes. An algorithm is written that implements the method of iterative extensions in matrix form. In this algorithm, the iterative parameters are automatically selected and the stopping criterion is indicated when the estimate of the required accuracy is reached. Examples of application of the method of iterative extensions for solving problems on a computer are given.

Keywords: Poisson's equation; method of iterative extensions.

Introduction

Consider a mixed boundary value problem with homogeneous Dirichlet and Neumann boundary conditions for the Poisson equation in a bounded domain. The main problems in solving elliptic problems are usually associated with the complexity of the geometry of the domain, the height of the order of the differential equation, and the presence of Dirichlet boundary conditions [1–5]. We will proceed from the desired provisions that the proposed numerical methods should be stable to computational rounding errors, be asymptotically optimal in terms of computational costs, be sufficiently universal and have an easy implementation when calculating on a computer. To fulfill the indicated provisions when solving the problem under consideration, we will apply the method of iterative extensions as a generalization of the method of fictitious components [4–7]. Note that to solve the problem in a rectangular area, to the solution of which the solution of the original problem will be reduced, it is possible to use marching methods that are optimal in terms of costs [8–10].

1. Boundary problem

Let the first bounded flat area be given and the second bounded flat area selected.

$$\omega \in \{1, \mathrm{II}\},\ \Omega_\omega \subset \mathbb{R}^2.$$

It is required that the intersection of these regions is empty, and the union of their closures is a closure of the rectangular region.

$$\Omega_1 \cap \Omega_{\mathrm{II}} = \varnothing,\ \bar{\Omega}_1 \cup \bar{\Omega}_{\mathrm{II}} = \bar{\Pi}.$$

For each of these three areas, the boundary consists of a closure of the union of two open non-intersecting parts.

$$\partial\Pi = \bar{s},\ s = \Gamma_1 \cup \Gamma_2,\ \Gamma_1 \cap \Gamma_2 = \varnothing,$$

$$\partial\Omega_\omega = \bar{s}_\omega,\ s_\omega = \Gamma_{\omega,1} \cup \Gamma_{\omega,2},\ \Gamma_{\omega,1} \cap \Gamma_{\omega,2} = \varnothing.$$

We assume that the intersection of the boundaries of the first and second areas is the closure of the non-empty intersection of the first part of the boundary of the first area with the second part of the boundary of the second area.

$$\partial\Omega_{\text{I}} \cap \partial\Omega_{\text{II}} = \overline{S},\ S = \Gamma_{1,1} \cap \Gamma_{\text{II},2} \neq \varnothing.$$

All the considered parts of the boundaries for all domains are the union of a finite number of disjoint open arcs of sufficiently smooth curves. Areas of the boundary that do not have self-intersections and self-tangencies are considered.

In the first domain, we consider a mixed boundary value problem for the Poisson equation. In the second domain, we introduce a mixed boundary value problem for the homogeneous screened Poisson equation. On the first parts of the boundaries of the regions, we set the homogeneous Diri-Khleb condition. On the second parts of the boundaries of the regions, we consider the homogeneous Neumann condition. The problem in the first area is a solvable problem. We use the problem on the second domain as a zero fictitious continuation of the problem being solved. Here is the problem to be solved and the fictitious problem.

$$\Delta\breve{u}_\omega + \kappa_\omega\breve{u}_\omega = \breve{f}_\omega,\ \kappa_{\text{I}} = 0,\ \kappa_{\text{II}} \geq 0,\ \breve{f}_{\text{II}} = 0,$$

$$\breve{u}_\omega\Big|_{\Gamma_{\omega,1}} = 0,\ \frac{\partial\breve{u}_\omega}{\partial n}\Big|_{\Gamma_{\omega,2}} = 0. \tag{1}$$

Consider the problem to be solved and the fictitious problem in variational form as the problem of representing linear functionals in the form of scalar products in functional spaces.

$$\breve{u}_\omega \in \breve{H}_\omega:\ \text{A}_\omega(\breve{u}_\omega,\breve{v}_\omega) = F_\omega(\breve{v}_\omega)\ \forall\breve{v}_\omega \in \breve{H}_\omega. \tag{2}$$

The spaces of solutions for such problems will be the following spaces of Sobolev functions.

$$\breve{H}_\omega = \breve{H}_\omega(\Omega_\omega) = \left\{\breve{v}_\omega \in W_2^1(\Omega_\omega) : \breve{v}\big|_{\Gamma_{\omega,1}} = 0\right\}.$$

The right-hand sides of these problems are linear functionals.

$$F_\omega(\breve{v}_\omega) = (\breve{f}_\omega,\breve{v}_\omega),\ (\breve{f}_\omega,\breve{v}_\omega) = \int_{\Omega_\omega} \breve{f}_\omega\breve{v}_\omega d\Omega_\omega.$$

The left-hand sides of these problems have bilinear forms.

$$\text{A}_\omega(\breve{u}_\omega,\breve{v}_\omega) = \int_{\Omega_\omega} (\breve{u}_{\omega x}\breve{v}_{\omega y} + \breve{u}_{\omega y}\breve{v}_{\omega y} + \kappa_\omega\breve{u}_\omega\breve{v}_\omega)d\Omega_\omega.$$

We assume that bilinear forms define in the spaces of solutions of the problems under consideration normalizations equivalent to those of the corresponding Sobolev spaces.

$$\exists c_1, c_2 > 0 : c_1\left\|\breve{v}_\omega\right\|^2_{W_2^1(\Omega_\omega)} \leq \text{A}_\omega(\breve{v}_\omega,\breve{v}_\omega) \leq c_2\left\|\breve{v}_\omega\right\|^2_{W_2^1(\Omega_\omega)}\ \ \forall\breve{v}_\omega \in \breve{H}_\omega.$$

These assumptions ensure the existence and uniqueness of the solution for each of the problems under consideration [1]. Note that the solution to the fictitious problem is zero.

2. Continued problem in operator form

It is possible to jointly consider the solved and fictitious problems in a variational form and an operator form. This problem will be called the continued problem.

$$\breve{u} \in \breve{V}:\ \text{A}_{\text{I}}(\breve{u}, I_1\breve{v}) + \text{A}_{\text{II}}(\breve{u},\breve{v}) = F_{\text{I}}(I_1\breve{v})\ \forall\breve{v} \in \breve{V},$$

$$\breve{u} \in \breve{V}:\ \breve{B}\breve{u} = \breve{f}, \tag{3}$$

if you define the operator and the right side in the previous problem as follows

$$(\breve{B}\breve{u},\breve{v}) = \text{A}_{\text{I}}(\breve{u}, I_1\breve{v}) + \text{A}_{\text{II}}(\breve{u},\breve{v})\ \forall\breve{u},\breve{v} \in \tilde{V},\ (\breve{f},\breve{v}) = F_{\text{I}}(I_1\breve{v})\ \forall\breve{v} \in \tilde{V}.$$

$$F(\breve{v}) = (\breve{f},\breve{v}),\ (\breve{f},\breve{v}) = \int_{\Pi} \breve{f}\breve{v}d\Pi.$$

The extended space of solutions for such a problem will be the following space of Sobolev functions

$$\breve{V} = \breve{V}(\Pi) = \left\{\breve{v} \in W_2^1(\Pi) : \breve{v}\big|_{\Gamma_1} = 0\right\}.$$

In the extended solution space, there is a subspace that is the solution space of the continued problem. This is the space of solutions to the original problem in the first region, padded by zero to the rest of the rectangular region.

$$\breve{V}_1 = \breve{V}_1(\Pi) = \left\{ \breve{v}_1 \in \breve{V} : \breve{v}_1\big|_{\Pi \setminus \Omega_1} = 0 \right\}.$$

In the formulation of the continued problem, we use the projection operator from the extended solution space of the continued problem to the space of its solutions.

$$I_1 : \breve{V} \mapsto \breve{V}_1,\ \breve{V}_1 = im I_1,\ I_1 = I_1^2.$$

Additionally, we introduce subspaces in the extended solution space.

$$\breve{V}_3 = \breve{V}_3(\Pi) = \left\{ \breve{v}_3 \in \breve{V} : \breve{v}_3\big|_{\Pi \setminus \Omega_{\mathrm{II}}} = 0 \right\},$$

$$\breve{V}_2 = \breve{V}_2(\Pi) = \left\{ \breve{v}_2 \in \breve{V} : \mathrm{A}(\breve{v}_2, \breve{v}_1) = 0\ \forall \breve{v}_1 \in \breve{V}_1,\ \mathrm{A}(\breve{v}_2, \breve{v}_3) = 0\ \forall \breve{v}_3 \in \breve{V}_3 \right\},\ \breve{V} = \breve{V}_1 \oplus \breve{V}_{\mathrm{II}},\ \breve{V}_{\mathrm{II}} = \breve{V}_2 \oplus \breve{V}_3.$$

We used a bilinear form, which is the sum of bilinear forms.

$$\mathrm{A}(\breve{u}, \breve{v}) = \mathrm{A}_1(\breve{u}, \breve{v}) + \mathrm{A}_{\mathrm{II}}(\breve{u}, \breve{v})\ \forall \breve{u}, \breve{v} \in \breve{V}.$$

We assume that the bilinear form defines in the extended solution space a normalization equivalent to the normalization of the corresponding Sobolev space.

$$\exists c_1, c_2 > 0 : c_1 \|\breve{v}\|^2_{W_2^1(\Pi)} \le \mathrm{A}(\breve{v}, \breve{v}) \le c_2 \|\breve{v}\|^2_{W_2^1(\Pi)}\ \forall \breve{v} \in \breve{V}.$$

We assume that the Sobolev spaces used are such that the continuation of functions with preservation of the norm is possible in them. We will use this provision, which is usual in such cases, in the indicated form.

$$\exists \breve{\beta}_1 \in (0;1],\ \breve{\beta}_2 \in [\breve{\beta}_1;1] : \breve{\beta}_1 \mathrm{A}(\breve{v}_2, \breve{v}_2) \le \mathrm{A}_{\mathrm{II}}(\breve{v}_2, \breve{v}_2) \le \breve{\beta}_2 \mathrm{A}(\breve{v}_2, \breve{v}_2)\ \forall \breve{v}_2 \in \breve{V}_2,$$

$$\exists \breve{\beta}_1 \in (0;1],\ \breve{\beta}_2 \in [\breve{\beta}_1;1] : \breve{\beta}_1 (\breve{\mathrm{A}} \breve{v}_2, \breve{v}_2) \le (\breve{\mathrm{A}}_{\mathrm{II}} \breve{v}_2, \breve{v}_2) \le \breve{\beta}_2 (\breve{\mathrm{A}} \breve{v}_2, \breve{v}_2)\ \forall \breve{v}_2 \in \breve{V}_2,$$

where the operators under consideration are defined as follows

$$\breve{\mathrm{A}} = \breve{\mathrm{A}}_{\mathrm{I}} + \breve{\mathrm{A}}_{\mathrm{II}},\ (\breve{\mathrm{A}} \breve{u}, \breve{v}) = \mathrm{A}(\breve{u}, \breve{v}),\ (\breve{\mathrm{A}}_{\mathrm{I}} \breve{u}, \breve{v}) = \mathrm{A}_1(\breve{u}, \breve{v}),\ (\breve{\mathrm{A}}_{\mathrm{II}} \breve{u}, \breve{v}) = \mathrm{A}_{\mathrm{II}}(\breve{u}, \breve{v})\ \forall \breve{u}, \breve{v} \in \breve{V}.$$

In this case, the continued problem has a unique solution. The solution to the continued problem is the solution to the original problem in the first area, continued by zero to the rest of the rectangular area. Note that the solution to the original problem and the solution to the original problem continued by zero, that is, the solution to the continued problem can be denoted in the same way as a function and its continuation.

3. Method of iterative extensions in operator form

Let us present a modified method of fictitious components in a variational form and an operator form. Consider the iterative process at each step, which we solve the extended problem with a bilinear form from the continued problem, but without the projection operator. We seek the solution to this problem in the extended solution space of the continued problem, in the solution space of the extended problem.

$$\breve{u}^k \in \breve{V} :\ \mathrm{A}(\breve{u}^k - \breve{u}^{k-1}, \breve{v}) = -\tau_{k-1}(\mathrm{A}_1(\breve{u}^{k-1}, I_1 \breve{v}) + \mathrm{A}_{\mathrm{II}}(\breve{u}^{k-1}, \breve{v}) - F_1(I_1 \breve{v})),\ k \in \mathbb{N}\ \forall \breve{v} \in \breve{V},$$

$$\breve{u}^k \in \breve{V} :\ \breve{\mathrm{A}}(\breve{u}^k - \breve{u}^{k-1}) = -\tau_{k-1}(\breve{B} \breve{u}^{k-1} - \breve{f}),\ k \in \mathbb{N}, \tag{4}$$

$$\forall \breve{u}^0 \in \breve{V}_1,\ \tau_0 = 1.\ \tau_{k-1} = 2 \big/ (\breve{\beta}_1 + \breve{\beta}_2),\ k \in \mathbb{N} \setminus \{1\}.$$

Let us propose the method of iterative extensions as a generalization of the method of fictitious components. Consider an iterative process at each step of which we solve an extended problem with a bilinear form, this operator generated by a bilinear form.

$$C(\breve{u}, \breve{v}) = \mathrm{A}_1(\breve{u}, \breve{v}) + \breve{\gamma} \mathrm{A}_{\mathrm{II}}(\breve{u}, \breve{v})\ \forall \breve{u}, \breve{v} \in \breve{V},\ \breve{\gamma} \in (0; +\infty),$$

$$(\breve{C} \breve{u}, \breve{v}) = C(\breve{u}, \breve{v})\ \forall \breve{u}, \breve{v} \in \breve{V}.$$

We seek the solution to this problem in the extended solution space of the continued problem, in the solution space of the extended problem.

$$\breve{u}^k \in \breve{V} :\ C(\breve{u}^k - \breve{u}^{k-1}, \breve{v}) = -\tau_{k-1}(\mathrm{A}_1(\breve{u}^{k-1}, I_1 \breve{v}) + \mathrm{A}_{\mathrm{II}}(\breve{u}^{k-1}, \breve{v}) - F_1(I_1 \breve{v})),\ k \in \mathbb{N}\ \forall \breve{v} \in \breve{V},$$

$$\breve{u}^k \in \breve{V}: \ \breve{C}(\breve{u}^k - \breve{u}^{k-1}) = -\tau_{k-1}(\breve{B}\breve{u}^{k-1} - \breve{f}), \ k \in \mathbb{N}, \tag{5}$$

$$\forall \breve{u}^0 \in \breve{V}_1, \ \breve{\gamma} > \breve{\alpha}, \ \tau_0 = 1, \ \tau_{k-1} = (\breve{r}^{k-1}, \breve{\eta}^{k-1}) / (\breve{\eta}^{k-1}, \breve{\eta}^{k-1}), \ k \in \mathbb{N} \setminus \{1\}.$$

To calculate the iterative parameters, it is necessary to calculate the residuals, corrections and equivalent residuals, respectively.

$$\breve{r}^{k-1} = \breve{B}\breve{u}^{k-1} - \breve{f}, \ \breve{w}^{k-1} = \breve{C}^{-1}\breve{r}^{k-1}, \ \breve{\eta}^{k-1} = \breve{B}\breve{w}^{k-1}, \ k \in \mathbb{N}.$$

We will assume that the assumptions about the continuation of functions are fulfilled, which we now write in the following form

$$\exists \breve{\delta}_1 \in (0;+\infty), \ \breve{\delta}_2 \in [\breve{\delta}_1;+\infty): \breve{\delta}_1^2(\breve{C}\breve{v}_2, \breve{C}\breve{v}_2) \le (\breve{\mathrm{A}}_{\mathrm{II}}\breve{v}_2, \breve{\mathrm{A}}_{\mathrm{II}}\breve{v}_2) \le \breve{\delta}_2^2(\breve{C}\breve{v}_2, \breve{C}\breve{v}_2) \ \forall \breve{v}_2 \in \breve{V}_2,$$

$$\exists \breve{\alpha} \in (0;+\infty): (\breve{\mathrm{A}}_{\mathrm{I}}\breve{v}_2, \breve{\mathrm{A}}_{\mathrm{I}}\breve{v}_2) \le \breve{\alpha}^2(\breve{\mathrm{A}}_{\mathrm{II}}\breve{v}_2, \breve{\mathrm{A}}_{\mathrm{II}}\breve{v}_2) \ \forall \breve{v}_2 \in \breve{V}_2,$$

if the operators under consideration are defined in this way

$$\breve{C} = \breve{\mathrm{A}}_{\mathrm{I}} + \breve{\gamma}\breve{\mathrm{A}}_{\mathrm{II}}, \ (\breve{C}\breve{u}, \breve{v}) = C(\breve{u}, \breve{v}), \ (\breve{\mathrm{A}}_{\mathrm{I}}\breve{u}, \breve{v}) = \mathrm{A}_1(\breve{u}, \breve{v}), \ (\breve{\mathrm{A}}_{\mathrm{II}}\breve{u}, \breve{v}) = \mathrm{A}_{\mathrm{II}}(\breve{u}, \breve{v}) \ \forall \breve{u}, \breve{v} \in \breve{V}.$$

4. Continued problem in operator form on a finite-dimensional subspace

Let us introduce a finite-dimensional approximating subspace. Let's set the previously introduced rectangular area and parts of its border in a rectangular coordinate system.

$$\Pi = (0;b_1) \times (0;b_2), \ \Gamma_1 = \{b_1\} \times (0;b_2) \cup (0;b_1) \times \{b_2\}, \ \Gamma_2 = \{0\} \times (0;b_2) \cup (0;b_1) \times \{0\}, \ b_1, b_2 \in (0;+\infty).$$

In this rectangular area, on the second part of its border and at the origin, we will introduce a grid.

$$(x_i; y_j) = ((i-1)h_1; (j-1)h_2), \ h_1 = b_1/m, \ h_2 = b_2/n, \ i = 1,2...,m, \ j = 1,2...,n, \ m,n \in \mathbb{N}.$$

Consider grid functions with values at the nodes of the introduced grid.

$$v_{i,j} = v(x_i; y_j) \in \mathbb{R}, \ i = 1,2...,m, \ j = 1,2...,n, \ m,n \in \mathbb{N}.$$

To complete the grid functions, we use bilinear basis functions.

$$\Phi^{i,j}(x;y) = \Psi^{1,i}(x)\Psi^{2,j}(y), \ i = 1,2...,m, \ j = 1,2...,n, \ m,n \in \mathbb{N},$$

$$\Psi^{1,i}(x) = \Psi(x/h_1 - i + 1), \ \Psi^{2,i}(y) = \Psi(y/h_2 - j + 1),$$

$$\Psi(z) = \begin{cases} z, & z \in [0;1], \\ 2 - z, & z \in [1;2], \\ 0, & z \notin [0;2]. \end{cases}$$

We assume that the values of the basis functions outside the given rectangular area are equal to zero.

$$\Phi^{i,j}(x;y) = 0, \ (x;y) \notin \Pi, \ i = 1,2...,m, \ j = 1,2...,n, \ m,n \in \mathbb{N}.$$

Linear combinations of basis functions form a finite-dimensional subspace in the solution space of the extended problem.

$$\tilde{V} = \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} v_{i,j} \Phi^{i,j}(x;y) \right\} \subset \breve{V}.$$

Consider the continued problem in variational form and in operator form when the space of its solutions is replaced by its previously introduced finite-dimensional space.

$$\tilde{u} \in \tilde{V}: \ \mathrm{A}_1(\tilde{u}, I_1\tilde{v}) + \mathrm{A}_{\mathrm{II}}(\tilde{u}, \tilde{v}) = F_1(I_1\tilde{v}) \ \forall \tilde{v} \in \tilde{V},$$

$$\tilde{u} \in \tilde{V}: \ \tilde{B}\tilde{u} = \tilde{f}, \tag{6}$$

where we define the operator and the right-hand side in the previous problem as follows

$$(\tilde{B}\tilde{u}, \tilde{v}) = \mathrm{A}_1(\tilde{u}, I_1\tilde{v}) + \mathrm{A}_{\mathrm{II}}(\tilde{u}, \tilde{v}) \ \forall \tilde{u}, \tilde{v} \in \tilde{V}, \ (\tilde{f}, \tilde{v}) = F_1(I_1\tilde{v}) \ \forall \tilde{v} \in \tilde{V}.$$

In the now used finite-dimensional space, there is a finite-dimensional subspace of solutions for the continued problem. This is a finite-dimensional space of solutions to the original problem in the first region, padded by zero to the rest of the rectangular region.

$$\tilde{V}_1 = \tilde{V}_1(\Pi) = \left\{ \tilde{v}_1 \in \tilde{V} : \tilde{v}_1\big|_{\Pi \setminus \Omega_1} = 0 \right\}.$$

We assume that the projection operator acts similarly on the corresponding finite-dimensional subspaces. In this case, we assume that the operator of projection onto the solution space of the continued problem zeroes out all the coefficients of the basis functions of the supports, which do not lie completely in the first domain.

$$I_1 : \tilde{V} \mapsto \tilde{V}_1, \tilde{V}_1 = imI_1, I_1 = I_1^2.$$

In addition, we introduce the corresponding finite-dimensional subspaces.

$$\tilde{V}_3 = \tilde{V}_3(\Pi) = \left\{ \tilde{v}_3 \in \tilde{V} : \tilde{v}_3\big|_{\Pi \setminus \Omega_{II}} = 0 \right\},$$

$$\tilde{V}_2 = \tilde{V}_2(\Pi) = \left\{ \tilde{v}_2 \in \tilde{V} : \mathrm{A}(\tilde{v}_2, \tilde{v}_1) = 0 \ \forall \tilde{v}_1 \in \tilde{V}_1,\ \mathrm{A}(\tilde{v}_2, \tilde{v}_3) = 0 \ \forall \tilde{v}_3 \in \tilde{V}_3 \right\}, \tilde{V} = \tilde{V}_1 \oplus \tilde{V}_2 \oplus \tilde{V}_3.$$

We will assume that for finite-dimensional subspaces approximating the corresponding spaces, the assumptions about the continuation of functions in the same form are fulfilled.

$$\exists \tilde{\beta}_1 \in (0;1],\ \tilde{\beta}_2 \in [\tilde{\beta}_1;1] : \tilde{\beta}_1 \mathrm{A}(\tilde{v}_2, \tilde{v}_2) \le \mathrm{A}_{II}(\tilde{v}_2, \tilde{v}_2) \le \tilde{\beta}_2 \mathrm{A}(\tilde{v}_2, \tilde{v}_2)\ \forall \tilde{v}_2 \in \tilde{V}_2,$$

$$\exists \tilde{\beta}_1 \in (0;1],\ \tilde{\beta}_2 \in [\tilde{\beta}_1;1] : \tilde{\beta}_1 (\tilde{\mathrm{A}}\tilde{v}_2, \tilde{v}_2) \le (\tilde{\mathrm{A}}_{II}\tilde{v}_2, \tilde{v}_2) \le \tilde{\beta}_2 (\tilde{\mathrm{A}}\tilde{v}_2, \tilde{v}_2)\ \forall \tilde{v}_2 \in \tilde{V}_2,$$

where the operators under consideration are defined as follows

$$\tilde{\mathrm{A}} = \tilde{\mathrm{A}}_I + \tilde{\mathrm{A}}_{II}, (\tilde{\mathrm{A}}\tilde{u}, \tilde{v}) = \mathrm{A}(\tilde{u}, \tilde{v}),\ (\tilde{\mathrm{A}}_I \tilde{u}, \tilde{v}) = \mathrm{A}_1(\tilde{u}, \tilde{v}),\ (\tilde{\mathrm{A}}_{II}\tilde{v}, \tilde{v}) = \mathrm{A}_{II}(\tilde{u}, \tilde{v})\ \forall \tilde{u}, \tilde{v} \in \tilde{V}.$$

Piecewise linear basis functions can be taken instead of bilinear basis functions.

5. The method of iterative extensions in operator form on a finite-dimensional subspace

Consider a modified method of fictitious components in variational form and operator form on a finite-dimensional subspace.

$$\tilde{u}^k \in \tilde{V} :\ \mathrm{A}(\tilde{u}^k - \tilde{u}^{k-1}, \tilde{v}) = -\tau_{k-1}(\mathrm{A}_1(\tilde{u}^{k-1}, I_1\tilde{v}) + \mathrm{A}_{II}(\tilde{u}^{k-1}, \tilde{v}) - F_1(I_1\tilde{v})),\ k \in \mathbb{N}\ \forall \tilde{v} \in \tilde{V},$$

$$\tilde{u}^k \in \tilde{V} :\ \tilde{\mathrm{A}}(\tilde{u}^k - \tilde{u}^{k-1}) = -\tau_{k-1}(\tilde{B}\tilde{u}^{k-1} - \tilde{f}),\ k \in \mathbb{N}, \tag{7}$$

$$\forall \tilde{u}^0 \in \tilde{V}_1,\ \tau_0 = 1.\ \tau_{k-1} = 2/(\tilde{\beta}_1 + \tilde{\beta}_2),\ k \in \mathbb{N} \setminus \{1\}.$$

We propose the method of iterative extensions as a generalization of the method of fictitious components on a finite-dimensional subspace. Consider an iterative process at each step of which we solve an extended problem with a bilinear form, with an operator generated by this bilinear form on a finite-dimensional subspace.

$$C(\tilde{u}, \tilde{v}) = \mathrm{A}_1(\tilde{u}, \tilde{v}) + \tilde{\gamma}\mathrm{A}_{II}(\tilde{u}, \tilde{v})\ \forall \tilde{u}, \tilde{v} \in \breve{V},\ \tilde{\gamma} \in (0; +\infty),$$

$$(\tilde{C}\tilde{u}, \tilde{v}) = C(\tilde{u}, \tilde{v})\ \forall \tilde{u}, \tilde{v} \in \tilde{V}.$$

We seek a solution to this problem in the extended solution space of the continued problem, in the solution space of the extended problem, but on a finite-dimensional subspace.

$$\tilde{u}^k \in \tilde{V} :\ C(\tilde{u}^k - \tilde{u}^{k-1}, \tilde{v}) = -\tau_{k-1}(\mathrm{A}_1(\tilde{u}^{k-1}, I_1\tilde{v}) + \mathrm{A}_{II}(\tilde{u}^{k-1}, \tilde{v}) - F_1(I_1\tilde{v})),\ k \in \mathbb{N}\ \forall \tilde{v} \in \tilde{V},$$

$$\tilde{u}^k \in \tilde{V} :\ \tilde{C}(\tilde{u}^k - \tilde{u}^{k-1}) = -\tau_{k-1}(\tilde{B}\tilde{u}^{k-1} - \tilde{f}),\ k \in \mathbb{N}, \tag{8}$$

$$\forall \tilde{u}^0 \in \tilde{V}_1,\ \tilde{\gamma} > \tilde{\alpha},\ \tau_0 = 1,\ \tau_{k-1} = (\tilde{r}^{k-1}, \tilde{\eta}^{k-1})/(\tilde{\eta}^{k-1}, \tilde{\eta}^{k-1}),\ k \in \mathbb{N} \setminus \{1\}.$$

To calculate the iterative parameters, it is necessary to calculate the residuals, corrections and equivalent residuals, respectively.

$$\tilde{r}^{k-1} = \tilde{B}\tilde{u}^{k-1} - \tilde{f},\ \tilde{w}^{k-1} = \tilde{C}^{-1}\tilde{r}^{k-1},\ \tilde{\eta}^{k-1} = \tilde{B}\tilde{w}^{k-1},\ k \in \mathbb{N}.$$

We will now assume that the assumptions about the continuation of functions are fulfilled, which we will write in the form.

$$\exists \tilde{\delta}_1 \in (0; +\infty),\ \tilde{\delta}_2 \in [\delta_1; +\infty) : \tilde{\delta}_1^2 (\tilde{C}\tilde{v}_2, \tilde{C}\tilde{v}_2) \le (\tilde{\mathrm{A}}_{II}\tilde{v}_2, \tilde{\mathrm{A}}_{II}\tilde{v}_2) \le \tilde{\delta}_2^2 (\tilde{C}\tilde{v}_2, \tilde{C}\tilde{v}_2)\ \forall \tilde{v}_2 \in \tilde{V}_2,$$

$$\exists \tilde{\alpha} \in (0; +\infty) : (\tilde{\mathrm{A}}_I\tilde{v}_2, \tilde{\mathrm{A}}_I\tilde{v}_2) \le \tilde{\alpha}^2 (\tilde{\mathrm{A}}_{II}\tilde{v}_2, \tilde{\mathrm{A}}_{II}\tilde{v}_2)\ \forall \tilde{v}_2 \in \tilde{V}_2,$$

where the operators under consideration are defined as follows.

$$\tilde{C} = \tilde{\mathrm{A}}_I + \tilde{\gamma}\tilde{\mathrm{A}}_{II},\ (\tilde{C}\tilde{u}, \tilde{v}) = C(\tilde{u}, \tilde{v}),\ (\tilde{\mathrm{A}}_I\tilde{u}, \tilde{v}) = \mathrm{A}_1(\tilde{u}, \tilde{v}),\ (\tilde{\mathrm{A}}_{II}\tilde{u}, \tilde{v}) = \mathrm{A}_{II}(\tilde{u}, \tilde{v})\ \forall \tilde{u}, \tilde{v} \in \tilde{V}.$$

6. The method of iterative extensions in matrix form

Consider the continued problem in matrix form. Approximating the continued problem using a finite-dimensional subspace, we obtain a linear system of algebraic equations.

$$\overline{u} \in \mathbb{R}^N :\ B\overline{u} = \overline{f},\ \overline{f} \in \mathbb{R}^N. \tag{9}$$

We also assume that the operator of projection onto the solution space of the continued problem zeroes out all the coefficients of the basis functions of the supports, which do not lie completely in the first domain. We get the continued problem in matrix form by defining the continued matrix and the continued right-hand side of the system.

$$\left\langle B\bar{u},\bar{v}\right\rangle = \mathrm{A}_1(\tilde{u},I_1\tilde{v}) + \mathrm{A}_{\mathrm{II}}(\tilde{u},\tilde{v})\ \forall\tilde{u},\tilde{v}\in\tilde{V},\ \left\langle \bar{f},\bar{v}\right\rangle = F_1(I_1\tilde{v})\ \forall\tilde{v}\in\tilde{V},$$

$$\left\langle \bar{f},\bar{v}\right\rangle = (\bar{f},\bar{v})h_1h_2 = \bar{f}\bar{v}h_1h_2,\ \bar{v} = (v_1,v_2,...,v_N)'\in\mathbb{R}^N,\ N = mn.$$

As an example, consider the numbering of grid nodes, coefficients of basis functions and corresponding basis functions.

$$v_{n(i-1)+j} = v_{i,j},\ \Phi_{n(i-1)+j} = \Phi^{i,j}(x_i;y_j),\ i=1,2...,m,\ j=1,2...,n,$$

$$\tilde{v} = \sum_{i=1}^{m}\sum_{j=1}^{n} v_{i,j}\Phi^{i,j}(x;y) = \sum_{l=1}^{N} v_l\Phi_l.$$

But we number from the beginning the basis functions of the carriers, which are completely contained in the first area. Further, we continue to number the basis functions of the carriers, which cross the border of the first region and the second region simultaneously. We complete the numbering on the basis of the support functions, which are completely contained in the second area. With this numbering, the arising vectors have the following form.

$$\bar{v} = (\bar{v}_1',\bar{v}_2',\bar{v}_3')',\ \bar{u} = (\bar{u}_1',\bar{0}'\,\bar{0}')',\ \ \bar{f} = (\bar{f}_1',\bar{0}'\,\bar{0}')'.$$

We calculate the elements of the matrix, the components of the vector on the right side of the reduced system.

$$b_{ij} = h_1^{-1}h_2^{-1}(\mathrm{A}_1(\Phi_i,I_1\Phi_j) + \mathrm{A}_{\mathrm{II}}(\Phi_i,\Phi_j)),\ f_i = h_1^{-1}h_2^{-1}F_1(I_1\Phi_i),\ i,j=1,2,...,N.$$

Consider the modified method of fictitious components in matrix form. By approximating the modified method of fictitious components in a variational form using a finite-dimensional subspace with the previously indicated projection operator, we obtain the well-known method of fictitious components in matrix form.

$$\bar{u}^k\in\mathbb{R}^N:\ \mathrm{A}(\bar{u}^k-\bar{u}^{k-1}) = -\tau_{k-1}(B\bar{u}^{k-1}-\bar{f}),\ k\in\mathbb{N},$$

$$\forall\bar{u}^0\in\bar{V}_1,\ \tau_0=1.\ \tau_{k-1} = 2/(\tilde{\beta}_1+\tilde{\beta}_2),\ k\in\mathbb{N}\setminus\{1\}. \quad (10)$$

At each step of this iterative process, we obtain an extended matrix problem with an extended matrix.

$$\left\langle \mathrm{A}\bar{u},\bar{v}\right\rangle = \mathrm{A}(\tilde{u},\tilde{v})\ \forall\tilde{u},\tilde{v}\in\tilde{V}.$$

We calculate the elements of this matrix.

$$a_{ij} = h_1^{-1}h_2^{-1}\mathrm{A}(\Phi_i,\Phi_j),\ i,j=1,2,...,N.$$

The resulting matrices have a well-known structure.

$$\mathrm{A} = \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} & 0 \\ \mathrm{A}_{21} & \mathrm{A}_{22} & \mathrm{A}_{23} \\ 0 & \mathrm{A}_{32} & \mathrm{A}_{33} \end{bmatrix},\ B = \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} & 0 \\ 0 & \mathrm{A}_{02} & \mathrm{A}_{23} \\ 0 & \mathrm{A}_{32} & \mathrm{A}_{33} \end{bmatrix}.$$

We use a subspace of vectors.

$$\bar{V}_1 = \left\{\bar{v} = (\bar{v}_1',\bar{v}_2',\bar{v}_3')'\in\mathbb{R}^N:\bar{v}_2=\bar{0},\ \bar{v}_3=\bar{0}\right\}.$$

Additionally, we introduce vector subspaces as the corresponding finite-dimensional subspaces introduced earlier.

$$\bar{V}_3 = \left\{\bar{v} = (\bar{v}_1',\bar{v}_2',\bar{v}_3')'\in\mathbb{R}^N:\bar{v}_1=\bar{0},\ \bar{v}_2=\bar{0}\right\},$$

$$\bar{V}_2 = \left\{\bar{v} = (\bar{v}_1',\bar{v}_2',\bar{v}_3')'\in\mathbb{R}^N:\mathrm{A}_{11}\bar{v}_1+\mathrm{A}_{12}\bar{v}_2=\bar{0},\ \mathrm{A}_{23}\bar{v}_2+\mathrm{A}_{33}\bar{v}_3=\bar{0}\right\},\ \mathbb{R}^N = \bar{V}_1\oplus\bar{V}_2\oplus\bar{V}_3.$$

Note that the fictitious component method solves the continued problem in matrix form.

$$B\bar{u} = \bar{f}, \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} & 0 \\ 0 & \mathrm{A}_{02} & \mathrm{A}_{23} \\ 0 & \mathrm{A}_{32} & \mathrm{A}_{33} \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{0} \\ \bar{0} \end{bmatrix} = \begin{bmatrix} \bar{f}_1 \\ \bar{0} \\ \bar{0} \end{bmatrix},$$

By solving the continued problem in matrix form, solutions are obtained, respectively, of the original problem in matrix form and the zero solution of the fictitious problem in matrix form.

$$\mathrm{A}_{11}\bar{u}_1 = \bar{f}_1, \begin{bmatrix} \mathrm{A}_{02} & \mathrm{A}_{23} \\ \mathrm{A}_{32} & \mathrm{A}_{33} \end{bmatrix} \begin{bmatrix} \bar{u}_2 \\ \bar{u}_3 \end{bmatrix} = \begin{bmatrix} \bar{0} \\ \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{u}_2 \\ \bar{u}_3 \end{bmatrix} = \begin{bmatrix} \bar{0} \\ \bar{0} \end{bmatrix}.$$

Let us introduce the norms generated by the identity matrix, the extended matrix, and its square.

$$\|\bar{v}\| = \sqrt{\langle \bar{v}, \bar{v} \rangle}\ , \|\bar{v}\|_{\mathrm{A}} = \sqrt{\langle \mathrm{A}\bar{v}, \bar{v} \rangle}\ , \|\bar{v}\|_{\mathrm{A}^2} = \sqrt{\langle \mathrm{A}^2\bar{v}, \bar{v} \rangle}\ \forall \bar{v} \in \mathbb{R}^N.$$

Lemma 1. *In the method of fictitious components (10), an estimate is performed.*

$$\left\|\bar{u}^1 - \bar{u}\right\|_{\mathrm{A}^2} \le 2\left\|\bar{u}^0 - \bar{u}\right\|_{\mathrm{A}^2}.$$

Proof. Let us introduce the notation for the error in the iterative process (10).

$$\bar{\psi}^k = \bar{u}^k - \bar{u},\ k \in \mathbb{N} \cup \{0\}.$$

From the iterative process, we obtain equalities.

$$\left(\mathrm{A}(\bar{\psi}^1 - \bar{\psi}^0)\right)^2 = \left(-\mathrm{A}_{11}\bar{\psi}_1^0\right)^2,\ \mathrm{A}\bar{\psi}^1\mathrm{A}\bar{\psi}^1 - 2\mathrm{A}\bar{\psi}^1\mathrm{A}\bar{\psi}^0 + \mathrm{A}\bar{\psi}^0\mathrm{A}\bar{\psi}^0 = \mathrm{A}_{11}\bar{\psi}_1^0\mathrm{A}_{11}\bar{\psi}_1^0.$$

Note that the inequality holds.

$$\mathrm{A}\bar{\psi}^0\mathrm{A}\bar{\psi}^0 \ge \mathrm{A}_{11}\bar{\psi}_1^0\mathrm{A}_{11}\bar{\psi}_1^0.$$

We get inequalities.

$$\mathrm{A}\bar{\psi}^1\mathrm{A}\bar{\psi}^1 - 2\mathrm{A}\bar{\psi}^1\mathrm{A}\bar{\psi}^0 \le 0,\ (\mathrm{A}\bar{\psi}^1\mathrm{A}\bar{\psi}^1)^2 \le (2\mathrm{A}\bar{\psi}^1\mathrm{A}\bar{\psi}^0)^2 \le 4(\mathrm{A}\bar{\psi}^1\mathrm{A}\bar{\psi}^1)(\mathrm{A}\bar{\psi}^0\mathrm{A}\bar{\psi}^0).$$

After cancellation, we obtain the following inequalities.

$$\mathrm{A}\bar{\psi}^1\mathrm{A}\bar{\psi}^1 \le 4\mathrm{A}\bar{\psi}^0\mathrm{A}\bar{\psi}^0,\ \left\|\bar{\psi}^1\right\|_{\mathrm{A}^2} \le 2\left\|\bar{\psi}^0\right\|_{\mathrm{A}^2},\ \left\|\bar{u}^1 - \bar{u}\right\|_{\mathrm{A}^2} \le 2\left\|\bar{u}^0 - \bar{u}\right\|_{\mathrm{A}^2}.$$

Lemma 2. *In the iterative process (10), the estimate holds.*

$$\left\|\bar{u}^1 - \bar{u}\right\|_{\mathrm{A}} \le d\left\|\bar{u}^1 - \bar{u}\right\|_{\mathrm{A}^2}$$

A positive value in the inequality is estimated as an asymptotic equality.

$$d \approx (\lambda_{1,1} + \kappa_{\mathrm{II}})^{1/2}\lambda_{1,1}^{-1},\ h_1, h_2 \to 0,\ \lambda_{1,1} = \pi^2(b_1^{-2} + b_2^{-2})/4.$$

Proof. Note that there is an inequality with a positive value.

$$\exists d > 0 : (\mathrm{A}\bar{\psi}^1, \bar{\psi}^1) \le d^2(\mathrm{A}\bar{\psi}^1, \mathrm{A}\bar{\psi}^1).$$

Let us obtain an upper bound for the indicated quantity in the inequality in the form of an asymptotic equality.

$$(\breve{\mathrm{A}}\breve{\psi}^1, \breve{\psi}^1) \le ((-\Delta + \kappa_{\mathrm{II}})\breve{\psi}^1, \breve{\psi}^1) = \sum_{i=1}^{+\infty}\sum_{j=1}^{+\infty}(\lambda_{i,j} + \kappa_{\mathrm{II}})c_{i,j}^2 = \sum_{i=1}^{+\infty}\sum_{j=1}^{+\infty}\lambda_{i,j}c_{i,j}^2 + \sum_{i=1}^{+\infty}\sum_{j=1}^{+\infty}\kappa_{\mathrm{II}}c_{i,j}^2 \le$$

$$\le \frac{1}{\lambda_{1,1}}\sum_{i=1}^{+\infty}\sum_{j=1}^{+\infty}\lambda_{i,j}^2c_{i,j}^2 + \frac{\kappa_{\mathrm{II}}}{\lambda_{1,1}^2}\sum_{i=1}^{+\infty}\sum_{j=1}^{+\infty}\lambda_{i,j}^2c_{i,j}^2 = \frac{\lambda_{1,1} + \kappa_{\mathrm{II}}}{\lambda_{1,1}^2}\sum_{i=1}^{+\infty}\sum_{j=1}^{+\infty}\lambda_{i,j}^2c_{i,j}^2 = \frac{\lambda_{1,1} + \kappa_{\mathrm{II}}}{\lambda_{1,1}^2}(\Delta\breve{\psi}^1, \Delta\breve{\psi}^1) \le (\breve{\mathrm{A}}\breve{\psi}^1, \breve{\mathrm{A}}\breve{\psi}^1).$$

Here we used the properties of solutions to the spectral problem.

$$\breve{\psi}^1 = \sum_{i=1}^{+\infty}\sum_{j=1}^{+\infty}c_{i,j}\breve{\varphi}_{i,j},\ (\breve{\varphi}_{i,j}, \breve{\varphi}_{i,j}) = 1,\ (\breve{\varphi}_{i,j}, \breve{\varphi}_{p,l}) = 0,\ (i;j) \ne (p;l),\ i, j, p, l \in \mathbb{N},$$

where

$$\breve{\varphi}_{i,j} \in V\left((0;b_1)\times(0;b_2)\right) : -\Delta\breve{\varphi}_{i,j} = \lambda_{i,j}\breve{\varphi}_{i,j},\ \breve{\varphi}_{i,j} \ne 0,\ \lambda_{i,j} = 0{,}25\pi^2\left((2i-1)b_1^{-2} + (2j-1)b_2^{-2}\right),\ i, j \in \mathbb{N}.$$

Theorem 1. *In the method of fictitious components (10), convergence estimates are satisfied.*

$$\left\|\bar{u}^k-\bar{u}\right\|_{\mathrm{A}} \le \varepsilon\left\|\bar{u}^0-\bar{u}\right\|_{\mathrm{A}^2}=\varepsilon\left\|\bar{f}^0-\bar{f}\right\|,$$
$$\varepsilon = cq^{k-1},\ c=2d\in(0+\infty),\ k\in\mathbb{N},\ \bar{f}^0=\mathrm{A}\bar{u}^0,\ 0\le q=(\beta_2-\beta_1)/(\beta_1+\beta_2)<1,$$
$$d\approx(\lambda_{1,1}+\kappa_{\mathrm{II}})^{1/2}\lambda_{1,1}^{-1},\ h_1,h_2\to 0,\ \lambda_{1,1}=\pi^2(b_1^{-2}+b_2^{-2})/4.$$

In the method of fictitious components, the absolute error in the energy norm converges with the speed of a geometric progression.

Let us propose the method of iterative extensions as a generalization of the method of fictitious components in a matrix form. To solve the old problem (9), we will apply a new method. Let us define the matrices that we will use in what follows.

$$\left\langle \mathrm{A}_{\mathrm{I}}\bar{u},\bar{v}\right\rangle=\mathrm{A}_1(\tilde{u},\tilde{v}),\ \left\langle \mathrm{A}_{\mathrm{II}}\bar{u},\bar{v}\right\rangle=\mathrm{A}_{\mathrm{II}}(\tilde{u},\tilde{v})\ \forall\tilde{u},\tilde{v}\in\tilde{V}.$$

The introduced two matrices have a definite structure.

$$\mathrm{A}_{\mathrm{I}}=\begin{bmatrix}\mathrm{A}_{11} & \mathrm{A}_{12} & 0\\ \mathrm{A}_{21} & \mathrm{A}_{20} & 0\\ 0 & 0 & 0\end{bmatrix},\ \mathrm{A}_{\mathrm{II}}=\begin{bmatrix}0 & 0 & 0\\ 0 & \mathrm{A}_{02} & \mathrm{A}_{23}\\ 0 & \mathrm{A}_{32} & \mathrm{A}_{33}\end{bmatrix}.$$

Let us now define in a different way the extended matrix as the sum of the first matrix with the second matrix multiplied by an additional positive parameter.

$$C=\mathrm{A}_{\mathrm{I}}+\gamma\mathrm{A}_{\mathrm{II}},\ \begin{bmatrix}C_{11} & C_{12} & 0\\ C_{21} & C_{22} & C_{23}\\ 0 & C_{32} & C_{33}\end{bmatrix}=\begin{bmatrix}\mathrm{A}_{11} & \mathrm{A}_{12} & 0\\ \mathrm{A}_{21} & \mathrm{A}_{20} & 0\\ 0 & 0 & 0\end{bmatrix}+\gamma\begin{bmatrix}0 & 0 & 0\\ 0 & \mathrm{A}_{02} & \mathrm{A}_{23}\\ 0 & \mathrm{A}_{32} & \mathrm{A}_{33}\end{bmatrix},\ \gamma\in(0;+\infty).$$

We will assume that for finite-dimensional spaces that approximate the corresponding spaces, the previous assumptions about the continuation of functions are fulfilled, which we now write in matrix form.

$$\exists\delta_1\in(0;+\infty),\ \delta_2\in[\delta_1;+\infty):\ \delta_1^2\left\langle C\bar{v}_2,C\bar{v}_2\right\rangle\le\left\langle \mathrm{A}_{\mathrm{II}}\bar{v}_2,\mathrm{A}_{\mathrm{II}}\bar{v}_2\right\rangle\le\delta_2^2\left\langle C\bar{v}_2,C\bar{v}_2\right\rangle\ \forall\bar{v}_2\in\bar{V}_2,$$
$$\exists\alpha\in(0;+\infty):\left\langle \mathrm{A}_{\mathrm{I}}\bar{v}_2,\mathrm{A}_{\mathrm{I}}\bar{v}_2\right\rangle\le\alpha^2\left\langle \mathrm{A}_{\mathrm{II}}\bar{v}_2,\mathrm{A}_{\mathrm{II}}\bar{v}_2\right\rangle\ \forall\bar{v}_2\in\bar{V}_2.$$

Now we apply the method of iterative extensions to solve problem (9), as a generalization of the method of fictitious components, using the introduction of an additional parameter in the extended matrix. Note that the method of fictitious components with a single value of this additional parameter is obtained from the method of iterative extensions, but without taking into account the choice of iterative parameters.

$$\bar{u}^k\in\mathbb{R}^N:\ C(\bar{u}^k-\bar{u}^{k-1})=-\tau_{k-1}(B\bar{u}^{k-1}-\bar{f}),\ k\in\mathbb{N},$$
$$\forall\bar{u}^0\in\bar{V}_1,\ \gamma>\alpha,\ \tau_0=1,\ \tau_{k-1}=\left\langle\bar{r}^{k-1},\bar{\eta}^{k-1}\right\rangle/\left\langle\bar{\eta}^{k-1},\bar{\eta}^{k-1}\right\rangle,\ k\in\mathbb{N}\setminus\{1\}. \tag{11}$$

To calculate the iterative parameters, it is necessary to calculate the residuals, corrections and equivalent residuals, respectively.

$$\bar{r}^{k-1}=B\bar{u}^{k-1}-\bar{f},\ \bar{w}^{k-1}=C^{-1}\bar{r}^{k-1},\ \eta^{k-1}=B\bar{w}^{k-1},\ k\in\mathbb{N}.$$

Let us introduce the norm generated by the square of the extended matrix.

$$\|\bar{v}\|_{C^2}=\sqrt{\left\langle C^2\bar{v},\bar{v}\right\rangle}\ \forall\bar{v}\in\mathbb{R}^N.$$

Lemma 3. In the method of iterative extensions (11), the estimate is satisfied.

$$\left\|\bar{u}^1-\bar{u}\right\|_{C^2}\le 2\left\|\bar{u}^0-\bar{u}\right\|_{C^2}.$$

Proof. Let us introduce the notation for the error in the iterative process (11).

$$\bar{\psi}^k=\bar{u}^k-\bar{u},\ k\in\mathbb{N}\cup\{0\}.$$

From the iterative process, we obtain equalities.

$$\left\langle C(\bar{\psi}^1-\bar{\psi}^0),C(\bar{\psi}^1-\bar{\psi}^0)\right\rangle=\left\langle -\mathrm{A}_{11}\bar{\psi}_1^0,-\mathrm{A}_{11}\bar{\psi}_1^0\right\rangle,$$
$$\left\langle C\bar{\psi}^1,C\bar{\psi}^1\right\rangle-2\left\langle C\bar{\psi}^1,\mathrm{A}\bar{\psi}^0\right\rangle+\left\langle C\bar{\psi}^0,C\bar{\psi}^0\right\rangle=\left\langle \mathrm{A}_{11}\bar{\psi}_1^0,\mathrm{A}_{11}\bar{\psi}_1^0\right\rangle.$$

Note that the inequality holds.

$$\left\langle C\overline{\psi}^0, C\overline{\psi}^0 \right\rangle \geq \left\langle \mathrm{A}_{11}\overline{\psi}_1^0, \mathrm{A}_{11}\overline{\psi}_1^0 \right\rangle.$$

We get inequalities.

$$\left\langle C\overline{\psi},^1 C\overline{\psi}^1 \right\rangle - 2\left\langle C\overline{\psi}^1, C\overline{\psi}^0 \right\rangle \leq 0, \left\langle C\overline{\psi}^1, C\overline{\psi}^1 \right\rangle^2 \leq 4\left\langle C\overline{\psi}^1, C\overline{\psi}^0 \right\rangle^2 \leq 4\left\langle C\overline{\psi}^1, C\overline{\psi}^1 \right\rangle \left\langle C\overline{\psi}^0, C\overline{\psi}^0 \right\rangle.$$

After cancellation, we obtain the following inequalities.

$$\left\langle C\overline{\psi}^1, C\overline{\psi}^1 \right\rangle \leq 4\left\langle C\overline{\psi}^0, C\overline{\psi}^0 \right\rangle, \left\|\overline{\psi}^1\right\|_{C^2} \leq 2\left\|\overline{\psi}^0\right\|_{C^2}, \left\|\overline{u}^1 - \overline{u}\right\|_{C^2} \leq 2\left\|\overline{u}^0 - \overline{u}\right\|_{C^2}.$$

Theorem 2. *In the method of iterative extensions (5), convergence estimates are satisfied.*

$$\left\|\breve{u}^k - \breve{u}\right\|_{\breve{C}^2} \leq \varepsilon \left\|\breve{u}^0 - \breve{u}\right\|_{\breve{C}^2}, \ \varepsilon = 2(\breve{\delta}_2/\breve{\delta}_1)(\breve{\alpha}/\breve{\gamma})^{k-1}, \ k \in \mathbb{N},$$

where the relative errors converge with a geometric progression rate in the norm stronger than the energy norm generated by the operator of the extended problem

$$\|\breve{v}\|_{\breve{C}^2} = \sqrt{(\breve{C}\breve{v}, \breve{C}\breve{v})} \ \forall \breve{v} \in \breve{V}.$$

Proof. From the iterative process, we obtain equalities for errors and residuals.

$$\breve{\psi}^k = \breve{\psi}^{k-1} - \tau_k \breve{C}^{-1}\breve{\mathrm{A}}_{\mathrm{II}}\breve{\psi}^{k-1}, \ \breve{r}^k = \breve{r}^{k-1} - \tau_k \breve{\mathrm{A}}_{\mathrm{II}}\breve{C}^{-1}\breve{r}^{k-1}, \ k \in \mathbb{N} \setminus 1.$$

We will minimize the residuals.

$$0 \leq \left(\breve{r}^k, \breve{r}^k\right) = \tau_k^2 \left(\breve{\mathrm{A}}_{\mathrm{II}}\breve{C}^{-1}\breve{r}^{k-1}, \breve{\mathrm{A}}_{\mathrm{II}}\breve{C}^{-1}\breve{r}^{k-1}\right) - 2\tau_k \left(\breve{\mathrm{A}}_{\mathrm{II}}\breve{C}^{-1}\breve{r}^{k-1}, \breve{r}^{k-1}\right) + \left(\breve{r}^{k-1}, \breve{r}^{k-1}\right).$$

We select the iterative parameters from the condition for minimizing the residuals.

$$\tau_{k-1} = \frac{\left(\breve{\mathrm{A}}_{\mathrm{II}}\breve{C}^{-1}\breve{r}^{k-1}, \breve{r}^{k-1}\right)}{\left(\breve{\mathrm{A}}_{\mathrm{II}}\breve{C}^{-1}\breve{r}^{k-1}, \breve{\mathrm{A}}_{\mathrm{II}}\breve{C}^{-1}\breve{r}^{k-1}\right)} = \frac{\left(\breve{r}^{k-1}, \breve{\eta}^{k-1}\right)}{\left(\breve{\eta}^{k-1}, \breve{\eta}^{k-1}\right)}.$$

Note the existence of equality.

$$\tau_{k-1} = \frac{\left(\breve{\mathrm{A}}_{\mathrm{II}}\breve{C}^{-1}\breve{r}^{k-1}, \breve{r}^{k-1}\right)}{\left(\breve{\mathrm{A}}_{\mathrm{II}}\breve{C}^{-1}\breve{r}^{k-1}, \breve{\mathrm{A}}_{\mathrm{II}}\breve{C}^{-1}\breve{r}^{k-1}\right)} = \frac{\left(\breve{\mathrm{A}}_{\mathrm{II}}\breve{w}^{k-1}, \breve{C}\breve{w}^{k-1}\right)}{\left(\breve{\mathrm{A}}_{\mathrm{II}}\breve{w}^{k-1}, \breve{\mathrm{A}}_{\mathrm{II}}\breve{w}^{k-1}\right)}.$$

Let us introduce the notation

$$\breve{\mathrm{A}}_{\mathrm{I}}\breve{w}^{k-1} = \breve{a}, \ \breve{\mathrm{A}}_{\mathrm{II}}\breve{w}^{k-1} = \breve{b}.$$

We establish the positivity of the selected iterative parameters.

$$\tau_k = \frac{\left(\breve{b}, \breve{a} + \breve{\gamma}\breve{b}\right)}{\left(\breve{b}, \breve{b}\right)} = \breve{\gamma} - \frac{\left(\breve{a}, \breve{b}\right)}{\left(\breve{b}, \breve{b}\right)} \geq \breve{\gamma} - \frac{\left(\breve{a}, \breve{a}\right)^{1/2}\left(\breve{b}, \breve{b}\right)^{1/2}}{\left(\breve{b}, \breve{b}\right)} \geq \breve{\gamma} - \frac{\left(\breve{a}, \breve{a}\right)^{1/2}}{\left(\breve{b}, \breve{b}\right)^{1/2}} \geq \breve{\gamma} - \breve{\alpha} > 0.$$

We present the scalar products of the residuals for the selected iterative parameters.

$$\left(\breve{r}^k, \breve{r}^k\right) = \left(\breve{r}^{k-1}, \breve{r}^{k-1}\right) - \frac{\left(\breve{\mathrm{A}}_{\mathrm{II}}\breve{C}^{-1}\breve{r}^{k-1}, \breve{r}^{k-1}\right)^2}{\left(\breve{\mathrm{A}}_{\mathrm{II}}\breve{C}^{-1}\breve{r}^{k-1}, \breve{\mathrm{A}}_{\mathrm{II}}\breve{C}^{-1}\breve{r}^{k-1}\right)}.$$

Write out the ratio of the squared residual norms at adjacent iterations.

$$q_k^2 = \frac{\left(\breve{r}^k, \breve{r}^k\right)}{\left(\breve{r}^{k-1}, \breve{r}^{k-1}\right)} = 1 - \frac{\left(\breve{\mathrm{A}}_{\mathrm{II}}\breve{C}^{-1}\breve{r}^{k-1}, \breve{r}^{k-1}\right)^2}{\left(\breve{\mathrm{A}}_{\mathrm{II}}\breve{C}^{-1}\breve{r}^{k-1}, \breve{\mathrm{A}}_{\mathrm{II}}\breve{C}^{-1}\breve{r}^{k-1}\right)\left(\breve{r}^{k-1}, \breve{r}^{k-1}\right)} =$$

$$= \frac{\left(\breve{\mathrm{A}}_{\mathrm{II}}\breve{w}^{k-1}, \breve{\mathrm{A}}_{\mathrm{II}}\breve{w}^{k-1}\right)\left(\breve{C}\breve{w}^{k-1}, \breve{C}\breve{w}^{k-1}\right) - \left(\breve{\mathrm{A}}_{\mathrm{II}}\breve{w}^{k-1}, \breve{C}\breve{w}^{k-1}\right)^2}{\left(\breve{\mathrm{A}}_{\mathrm{II}}\breve{w}^{k-1}, \breve{\mathrm{A}}_{\mathrm{II}}\breve{w}^{k-1}\right)\left(\breve{C}\breve{w}^{k-1}, \breve{C}\breve{w}^{k-1}\right)} = \frac{\left(\breve{b}, \breve{b}\right)\left(\breve{a} + \breve{\gamma}\breve{b}, \breve{a} + \breve{\gamma}\breve{b}\right) - \left(\breve{b}, \breve{a} + \breve{\gamma}\breve{b}\right)^2}{\left(\breve{b}, \breve{b}\right)\left(\breve{a} + \breve{\gamma}\breve{b}, \breve{a} + \breve{\gamma}\breve{b}\right)}.$$

We introduce the notation.

$$\left(\breve{a}, \breve{a}\right) = a, \ \left(\breve{b}, \breve{b}\right) = b, \ \left(\breve{a}, \breve{b}\right) = z,$$

then

$$q_k^2 = \frac{ab - z^2}{b(a+\breve{\gamma}^2 b + 2\breve{\gamma} z)} \le \max_{|z|\le\sqrt{ab}} q_k^2(z) = q_k^2\left(\frac{-a}{\breve{\gamma}}\right) = \frac{a}{\breve{\gamma}^2 b} \le \frac{\breve{\alpha}^2}{\breve{\gamma}^2} = q^2,$$

considering that

$$q_k^2 \ge 0, \left(q_k^2(z)\right)_z' = \frac{-2\breve{\gamma}(z+a/\breve{\gamma})(z+\breve{\gamma}b)}{b(a+\breve{\gamma}^2 b+2\breve{\gamma}z)^2}, -\breve{\gamma}b < \frac{a+\breve{\gamma}^2 b}{2\breve{\gamma}} < -\sqrt{ab} < -\frac{a}{\breve{\gamma}} < \sqrt{ab}.$$

This is how we establish inequalities.

$$\left(\breve{A}_{II}\breve{\psi}^k, \breve{A}_{II}\breve{\psi}^k\right) \le q^2\left(\breve{A}_{II}\breve{\psi}^{k-1}, \breve{A}_{II}\breve{\psi}^{k-1}\right), \left(\breve{A}_{II}\breve{\psi}^k, \breve{A}_{II}\breve{\psi}^k\right) \le q^{2(k-1)}\left(\breve{A}_{II}\breve{\psi}^1, \breve{A}_{II}\breve{\psi}^1\right), k \in \mathbb{N}\setminus\{1\}.$$

Considering that

$$\left\langle \breve{C}\breve{\psi}^k,\ \breve{C}\breve{\psi}^k\right\rangle \le \breve{\delta}_1^{-2}\left(\breve{A}_{II}\breve{\psi}^k, \breve{A}_{II}\breve{\psi}^k\right), \left(\breve{A}_{II}\breve{\psi}^1, \breve{A}_{II}\breve{\psi}^1\right) \le \breve{\delta}_2^2\left(\breve{C}\breve{\psi}^1,\ \breve{C}\breve{\psi}^1\right) \le 4\breve{\delta}_2^2\left(\breve{C}\breve{\psi}^0,\ \breve{C}\breve{\psi}^0\right),$$

$$\breve{\delta}_2^2\left(\breve{C}\breve{\psi}^1,\ \breve{C}\breve{\psi}^1\right) \le 4\breve{\delta}_2^2\left(\breve{C}\breve{\psi}^0,\ \breve{C}\breve{\psi}^0\right),$$

we obtain an inequality from which the convergence estimate in the method of iterative extensions follows. Here we take into account the passage to the limit in the inequality.

$$\left(\breve{C}\breve{\psi}^1, \breve{C}\breve{\psi}^1\right) \approx \left\langle C\overline{\psi}^1,\ C\overline{\psi}^1\right\rangle \le 4\left\langle C\overline{\psi}^0,\ C\overline{\psi}^0\right\rangle \approx 4\left(\breve{C}\breve{\psi}^0, \breve{C}\breve{\psi}^0\right),\ h_1, h_2 \to 0.$$

Theorem 3. *In the method of iterative extensions on a finite-dimensional subspace (8), convergence estimates are satisfied.*

$$\left\|\tilde{u}^k - \tilde{u}\right\|_{\tilde{C}^2} \le \varepsilon\left\|\tilde{u}^0 - \tilde{u}\right\|_{\tilde{C}^2},\ \varepsilon = 2(\tilde{\delta}_2/\tilde{\delta}_1)(\tilde{\alpha}/\tilde{\gamma})^{k-1},\ k \in \mathbb{N},$$

where the relative errors converge with a geometric progression rate in the norm stronger than the energy norm generated by the operator of the extended problem on the finite-dimensional and approximating subspace

$$\|\tilde{v}\|_{\tilde{C}^2} = \sqrt{(\tilde{C}\tilde{v}, \tilde{C}\tilde{v})}\ \ \forall \tilde{v} \in \tilde{V}.$$

We assume that the properties are fulfilled in the approximation.

$$(\tilde{A}_I\tilde{v}, \tilde{A}_I\tilde{v}) \approx (\breve{A}_I\breve{v}, \breve{A}_I\breve{v}),\ (\tilde{A}_{II}\tilde{v}, \tilde{A}_{II}\tilde{v}) \approx (\breve{A}_{II}\breve{v}, \breve{A}_{II}\breve{v}),\ h_1, h_2 \to 0.$$

In this case, the previous theorem is obtained from this theorem under the passage to the limit.

Theorem 4. *There are estimates in the method of iterative extensions in matrix form (11).*

$$\left\|\overline{u}^k - \overline{u}\right\|_{C^2} \le \varepsilon\left\|\overline{u}^0 - \overline{u}\right\|_{C^2},\ \varepsilon = 2(\delta_2/\delta_1)(\alpha/\gamma)^{k-1},\ k \in \mathbb{N},$$

where the relative errors converge with a geometric progression rate in the norm stronger than the energy norm generated by the operator of the extended problem

$$\|\overline{v}\|_{C^2} = \sqrt{(C\overline{v}, C\overline{v})}\ \ \forall \overline{v} \in \mathbb{R}^N.$$

Remark 1. *Passing to the limit from Theorems 3 and 4, Theorem 2 follows. Theorem 3 and Theorem 4 coincide up to notation. The proof of Theorem 4 is similar to the proof of Theorem 2 and does not use the passage to the limit in the inequality obtained at the first iteration.*

7. Algorithm that implements the method of iterative extensions in matrix form

We use the method of minimum residuals with a zero initial approximation.

I. Calculate the square of the norm of the initial absolute error.

$$E_0 = \left\langle \overline{f}, \overline{f}\right\rangle.$$

II. Find the first approximation.

$$\overline{u}^1 = C^{-1}\overline{f}.$$

III. We calculate the residual.

$$\overline{r}^{k-1} = B\overline{u}^{k-1} - \overline{f} = A_{II}\overline{u}^{k-1},\ k \in \mathbb{N}\setminus\{1\}.$$

IV. Calculate the square of the absolute error rate.

$$E_{k-1} = \left\langle \overline{r}^{k-1}, \overline{r}^{k-1}\right\rangle,\ k \in \mathbb{N}\setminus\{1\}.$$

V. Finding an amendment.

$$\overline{w}^{k-1} = C^{-1}\overline{r}^{k-1},\ k \in \mathbb{N}\setminus\{1\}.$$

VI. We calculate the equivalent residual.

$$\bar{\eta}^{k-1} = B\bar{w}^{k-1} = A_{\mathrm{II}}\bar{w}^{k-1},\ k \in \mathbb{N} \setminus \{1\}.$$

VII. We calculate the iteration parameter.

$$\tau_{k-1} = \left\langle \bar{r}^{k-1}, \bar{\eta}^{k-1} \right\rangle \Big/ \left\langle \bar{\eta}^{k-1}, \bar{\eta}^{k-1} \right\rangle,\ k \in \mathbb{N} \setminus \{1\}.$$

VIII. We calculate the next approximation.

$$\bar{u}^{k} = \bar{u}^{k-1} - \tau_{k-1}\bar{w}^{k-1},\ k \in \mathbb{N} \setminus \{1\}.$$

IX. Checking the condition for stopping iterations.

$$\mathrm{E}_{k-1} \le \mathrm{E}_0 \mathrm{E},\ k \in \mathbb{N} \setminus \{1\},\ \mathrm{E} = 0{,}0001 \in (0;1).$$

If the condition for stopping iterations is not met, then everything is repeated from point III.

8. Examples of application of the method of iterative extensions in matrix form

Consider tasks using the following areas.

$$\Pi = (0;6)\times(0;6),\ \Omega_{\mathrm{I}} = (0;6)\times(1;4), \Omega_{\mathrm{II}} = (0;6)\times(0;1) \cup (0;6)\times(4;6).$$

We assume that the areas have boundaries.

$$\Gamma_1 = (0;6)\times\{6\},\ \Gamma_2 = \{0,\ 6\}\times(0;6) \cup (0;6)\times\{0\},\ \Gamma_{1,1} = (0;6)\times\{1,\ 4\},\ \Gamma_{1,2} = \{0,\ 6\}\times(1;4),$$

$$\Gamma_{\mathrm{II},1} = (0;6)\times\{6\},\ \Gamma_{\mathrm{II},2} = (0;6)\times\{0,\ 1,\ 4\} \cup \{0,\ 6\}\times(0;1) \cup \{0,\ 6\}\times(4;6).$$

We select the right sides and the coefficient in the equations.

$$\breve{f}_1(x;y) = \begin{cases} 2,\ (x;y) \in (0;6)\times(1;1+h), \\ 0,\ (x;y) \in (0;6)\times(1+h;4), \end{cases}$$

$$\kappa_{\mathrm{II}}(x;y) = 2,\ (x;y) \in (0;6)\times(0;1),\ \kappa_{\mathrm{II}}(x;y) = 0,\ (x;y) \in (0;6)\times(4;6).$$

Here are the solutions to the problems.

$$\breve{u}_1(x;y) = \begin{cases} -y^2 - (h^2/3 - 2h - 2)y + h^2/3 - 2h - 1,\ (x;y) \in (0;6)\times(1;1+h], \\ (-h^2/3)y + 4h^2/3, \qquad (x;y) \in (0;6)\times[1+h;4). \end{cases}$$

When sampling, we select the grid steps. $h = h_1 = h_2 = 6/n,\ n = 6,12,...,102.$ In calculations by the method of iterative extensions with a zero initial approximation, the number of iterations is set at a predetermined estimate of the relative error.

$$k(\mathrm{E};n) = 8,\ n = 6,12,\ k(\mathrm{E};n) = 6,\ n = 18,...,102,\ \mathrm{E} = 0{,}0001.$$

Note that at the last iteration, on the finest grid, inequalities are satisfied.

$$\max_{(x_i;y_j)\in\Omega_1} \left(\left| u_{i,j}^2 - u_{i,j} \right| \Big/ \left| u_{i,j} \right| \right) \le 0{,}0022,\ \max_{(x_i;y_j)\in\Omega_1} \left| u_{i,j}^2 - u_{i,j} \right| \Big/ \max_{(x_i;y_j)\in\Omega_1} \left| u_{i,j} \right| \le 0{,}00043$$

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Received December 16, 2021

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Bulletin of the South Ural State University
Series "Mathematics. Mechanics. Physics"
2022, vol. 14, no. 1, pp. 64–76

УДК 519.63 **DOI: 10.14529/mmph220107**

АНАЛИЗ КРАЕВОЙ ЗАДАЧИ ДЛЯ УРАВНЕНИЯ ПУАССОНА

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Аннотация. Смешанная краевая задача для уравнения Пуассона рассматривается в ограниченной плоской области. Проводится продолжение этой задачи через границу с условием Дирихле до прямоугольной области. Предлагается рассмотрение продолженной задачи в операторном виде. Для решения продолженной задачи формулируется метод итерационных расширений в операторном виде. Продолженная задача в операторном виде рассматривается на конечномерном подпространстве. Для решения предыдущей задачи формулируется метод итерационных расширений в операторном виде на конечномерном подпространстве. Продолженная задача приводится в матричном виде. Для решения продолженной задачи в матричном виде формулируется метод итерационных расширений в матричном виде. Показывается, что в предложенных вариантах метода итерационных расширений относительные ошибки сходится в норме более сильной, чем энергетическая норма расширенной задачи со скоростью геометрической прогрессии. Итерационные параметры в указанных методах выбираются с помощью метода минимальных невязок. Указываются условия, достаточные для сходимости применяемых итерационных процессов. Выписан алгоритм, реализующий метод итерационных расширений в матричном виде. В данном алгоритме производится автоматический выбор итерационных параметров и указывается критерий остановки при достижении оценки требуемой точности. Приводятся примеры применения метода итерационных расширений для решения задач на ЭВМ.

Ключевые слова: уравнение Пуассона; метод итерационных расширений.

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Поступила в редакцию 16 декабря 2021 г.

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