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ON UNIQUENESS IN THE PROBLEMS OF DETERMINING POINT SOURCES IN MATHEMATICAL MODELS OF HEAT AND MASS TRANSFER

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Abstract. We consider the problem of determining point sources for mathematical models of heat and mass transfer. The values of a solution (concentrations) at some points lying inside the domain are taken as overdetermination conditions. A second-order parabolic equation is considered, on the right side of which there is a linear combination of the Dirac delta functions $\delta(x-x_i)$ with coefficients that depend on time and characterize the intensities of sources. Several different problems are considered, including the problem of determining the intensities of sources if their locations are given. In this case, we present the theorem of uniqueness of solutions, the proof of which is based on the Phragmén–Lindelöf theorem. Next, in the model case, we consider the problem of simultaneous determining the intensities of sources and their locations. The conditions on the number of measurements (the ovedetermination conditions) are described which ensure that a solution is uniquely determined. Examples are given to show the accuracy of the results. This problem arises when solving environmental problems, first of all, the problems of determining the sources of pollution in a water basin or atmosphere. The results are important when developing numerical algorithms for solving the problem. In the literature, such problems are solved numerically by reducing the problem to an optimal control problem and minimizing the corresponding objective functional. The examples show that this method is not always correct since the objective functional can have a significant number of minima.

Keywords: heat and mass transfer; parabolic equation; uniqueness; inverse problem; point source.

Introduction

Under consideration is the inverse problem of recovering the point sources in the model

$$u_t + Lu = \sum_{i=1}^{m} N_i(t)\delta(x-x_i) + f_0(t,x), Lu = -\Delta u + \sum_{i=1}^{n} a_i(x)u_{x_i} + a_0(x)u, \quad (1)$$

where $(x,t) \in Q = (0,T) \times G$, G is a domain in $\mathbb{R}^n$ $(n = 2,3)$ with boundary $\Gamma \in C^2$. The unknowns are the functions $N_i(t)$. The equation (1) is furnished with the initial and boundary conditions

$$Bu_S = g, u_{t=0} = u_0(x), S = (0,T) \times \Gamma, \quad (2)$$

where either $Bu = \frac{\partial u}{\partial \nu} + \sigma u$, or $Bu = u$ (ν is the outward unit normal to Γ), and the overdetermination conditions

$$u(y_j, t) = \psi_j(t), j = 1,2,\ldots,s. \quad (3)$$

These problems arise in mathematical modelling of heat and mass transfer processes, diffusion, filtration, and in many other fields (see [1–3]). In the theory of heat and mass transfer, the function u is the concentration of a transferred substance and the right part characterizes sources (sinks) [1]. In the most general formulation of the problem (1)–(3), the intensities $N_i(t)$ of point sources, their locations x_i and the number m are quantities to be determined. Some descriptions of models of this type can be found, for example, in [1]. A lot of articles are devoted to solving these inverse problems. The main results are connected with numerical methods of solving the problem and many of them are far from justified (see [4–16]). The problem is ill-posed and examples when the problem is not solvable or has many solutions are easily constructed. Very often the methods rely on reducing the problem to an optimal control problem and minimization of the corresponding objective functional [2, 4, 5, 9, 16]. However, it is possible that the corresponding functionals can have many local minima. Some theoretical results devoted to the problem (1)–(3) are available in [17–21]. The stationary case is treated in [20], where the Dirichlet data are complemented with the Neumann data and these data allow to solve the problem on recovering the number of sources, their locations, and intensities using test functions and a Prony-type algorithm. The model problem (1)–(3) ($G = R^n$) is considered in [21], where the explicit representation of solutions to the direct problem (the Poisson formula) and auxiliary variational problem are employed to determine numerically the quantities $\sum_i N_i r_{ij}^l$ (here $N_i(t) = \text{const}$ for all i and $r_{ij} = |x_i - y_j|$). The quantities found allow to determine the points $\{x_i\}$ and intensities N_i (see Theorem 2 and the corresponding algorithm in [21]). So the results of [21], for instance, say that the problem (1)–(3) and more general problem of simultaneous recovering points $\{x_i\}$ and intensities $\{N_i\}$ in some model situations is uniquely solvable. In the one-dimensional case uniqueness theorem for solutions to the problem (1)–(3) with $n = 1, m = 1$ is stated in [17]. Similar results are presented also in [22].

In this article the main attention is paid to uniqueness questions of solutions to the problem in some model cases and the general case as well. Examples showing the accuracy of the results obtained are displayed. The constructions can be used when developing numerical algorithms. The results are based on asymptotic representations of the Green functions of the corresponding elliptic problems (see [23]).

Preliminaries

First, we describe our conditions on the data and some corollaries of the results in [23]. Let G be a domain in $\mathbb{R}^n$. The symbols $L_p(G)$ and $W_p^s(G)$ ($1 \le p \le \infty$) stand for the Lebesgue and Sobolev spaces [24]. We also use the spaces $C^k(\overline{G})$ of k times differentiable functions (see the definitions in [24]). If Γ, S are some sets then the symbol $\rho(\Gamma, S)$ stands for the distance between these sets. The symbol $D(L)$ stands for the domain of an operator L. Denote by $B_r(x_0)$ the ball of radius r centered at x_0. Let $\vec{a} = (a_1, a_2)$ for $n = 2$ and $\vec{a} = (a_1, a_2, a_3)$ for $n = 3$. The brackets $(\cdot,\cdot)$ denote the inner product in R^n. Let

$$\psi(x) = \frac{1}{2}\int_0^1 (\vec{a}(x_0 + \tau(x - x_0)), (x - x_0))d\tau.$$

The coefficients in (1) are assumed to be real-valued and

$$a_i \in W_\infty^2(G)(i = 1, \ldots, n), \nabla\psi, \Delta\psi, a_0 \in L_\infty(G), \sigma \in C^1(\Gamma), \tag{4}$$

Consider the problem

$$-\Delta u + \sum_{i=1}^n a_i u_{x_i} + a_0 u + \lambda u = \delta(x - x_0), x \in G \subset \mathbb{R}^n, \tag{5}$$

$$Bu|_\Gamma = 0. \tag{6}$$

For the reader's convenience, here we present some results the article [23] (see Theorem 3.5, 3.9, 3.11, 3.12). We consider compact $K \subset G$, containing x_0, with properties: if $Bu = u$ and G is a domain with compact boundary then the convex hull of K is contained in G; if G is a domain with compact boundary and $Bu \ne u$ then $K \subset B_{\rho(x_0, \Gamma)}(x_0)$; if $G = \mathbb{R}^n$ or $G = \mathbb{R}_+^n$, then K is an arbitrary compact.

Theorem 1. [23]. Assume that the conditions (4) hold, K is a compact with the above properties, and if $G = \mathbb{R}_+^n$, then, in case $Bu \ne u$, $\sigma = 0$ (i.e., $Bu = u_{x_n}$) and $a_i \equiv 0$ for $i = 1, 2, \ldots, n$. Then there exists $\lambda_1 \ge 0$ such that for all $\lambda \ge \lambda_1$ a solution $u_n(x)$ $(n = 2, 3)$ to the problem (5), (6) in every domain $\{y \in K : 0 < \varepsilon \le |y - x_i|, i = 1, 2, \ldots, m\}$ admits the representation

$$u_2(x)=\frac{1}{2\sqrt{2\pi|x-x_0|}\lambda^{1/4}}e^{\psi(x)-\sqrt{\lambda}|x-x_0|}(1+O(\tfrac{1}{\sqrt{|\lambda|}}));\qquad(7)$$

$$u_3(x)=\frac{1}{4\pi|x-x_0|}e^{\psi(x)-\sqrt{\lambda}|x-x_0|}(1+O(\tfrac{1}{\sqrt{|\lambda|}})).\qquad(8)$$

Next theorem deals with solvability of the direct problem (1), (2). Let

$$u_0(x)\in W_2^1(G), u_0(x)|_\Gamma=g(x,0)\ if\ Bu=u.\qquad(9)$$

We also suppose that

$$f_0\in L_2(Q), g(x,t)\in W_2^{3/4,3/2}(S)\ if\ Bu=u, g(x,t)\in W_2^{1/4,1/2}(S)\ if\ Bu\neq u.\qquad(10)$$

Consider auxiliary problems

$$u_t+Lu=f_0(t,x), Bu_S=g, u_{t=0}=u_0(x),\qquad(11)$$

$$w_t+Lw=\sum_{i=1}^{m}N_i(t)\delta(x-x_i), Bw_S=0, w_{t=0}=0.\qquad(12)$$

Let $W_{p,B}^1(G)$ be a space of functions $u\in W_p^1(G)$ satisfying the homogeneous Dirichlet condition whenever $Bu=u$ and $W_{p,B}^1(G)=W_p^1(G)$ if $Bu\neq u$. Denote by $W_{p,B}^{-1}(G)$ the dual space to $W_{q,B}^1(G)$ (the duality is defined by the inner product in $L_2(G)$, see [25]).

The following theorem follows from [26], theorem 2 and [27], theorem 8.2.

Theorem 2. Let $T<\infty$ and let $p\in(1,n/(n-1))$. Assume that the conditions (9), (10) hold, $a_i\in L_\infty(G)$ $(i=0,1,\dots,n)$, and $N_i\in L_2(0,T)$ $(i=1,2,\dots,m)$. Then there exists a unique solution to the problem (1), (2) such that $u=w_0+w$, where $w_0\in W_2^{1,2}(Q)$ is a solution to the problem (11), w is a solution to the problem (12), $w\in L_2\left(0,\infty;W_{p,B}^1(G)\right)$, $w_t\in L_2\left(0,\infty;W_{p,B}^{-1}(G)\right)$ and $w\in W_2^{1,2}(Q_\varepsilon)$ with $Q_\varepsilon=\{(x,t)\in Q:|x-x_i|>\varepsilon\ \forall i\le m\}$ for all $\varepsilon>0$.

Main results

Here we present our uniqueness theorem for solutions to the problem (1)–(3). We introduce the functions

$$\varphi_j(x)=-\frac{1}{2}\int_0^1(\vec{a}(y_j+\tau(x-y_j)),(x-y_j))d\tau.$$

Let $\delta_j=\min_i r_{ij}, j=1,2,\dots,s$, where $r_{ij}=|x_i-y_j|$. Let A_0 be the matrix with entries $a_{ji}=e^{\varphi_j(x_i)}$ if $|x_i-y_j|=\delta_j$ and $a_{ji}=0$ otherwise. We assume, that:

$$\det A_0\neq 0\qquad(13)$$

Condition (4) is rewritten as follows: the coefficients of L are real-valued and

$$a_i\in W_\infty^2(G)(i=1,\dots,n),\nabla\varphi_j,\Delta\varphi_j,a_0\in L_\infty(G)(j\le s),\sigma\in C^1(\Gamma).\qquad(14)$$

Firstly, we justify uniqueness in the inverse problem (1)–(3) of recovering a solutions u and intensities N_i $(i=1,\dots,m)$. Points $\{x_i\}$ and their number are assumed to be known.

Theorem 3. Assume that $T<\infty$, $m=s$, and the conditions (13), (14) hold. Then a solution $(u,\vec{N})$ to the problem (1)–(3) such that u belongs to the class described in Theorem 2 and $N_i\in L_2(0,T)$ is unique.

Proof. It suffices to demonstrate that a solution to the problem (1)–(3) with homogeneous data is zero. In this case the auxiliary function $w_0=0$. Let a function u such that $u\in W_2^{1,2}(Q_\varepsilon)$ for any $\varepsilon>0$, $u\in L_2\left(0,T;W_{p,B}^1(G)\right)$, $u_t\in L_2\left(0,T;W_{p,B}^{-1}(G)\right)$ be a solution to the problem

$$u_t + Lu = \sum_{i=1}^{m} N_i(t)\delta(x - x_i), \tag{15}$$

$$Bu_S = 0, u_{t=0} = 0, \tag{16}$$

$$u(y_j, t) = 0,\ j = 1, 2, \ldots, s. \tag{17}$$

We integrate the equation (15) with respect to t and make the change of variables $w = \int_0^t u(\tau)d\tau$. This function is a solution to the problem

$$w_t + Lw = \sum_{i=1}^{m} s_i(t)\delta(x - x_i), s_i = \int_0^t N_i(\tau)d\tau \in W_2^1(0,T), s_i(0) = 0, \tag{18}$$

$$Bw_S = 0, w_{t=0} = 0, \tag{19}$$

$$w(y_j, t) = 0,\ j = 1, 2, \ldots, s. \tag{20}$$

Put $w = e^{\lambda t}v$, where $\lambda \in \mathbb{R}$. This function satisfies the equation

$$v_t + Lv + \lambda v = \sum_{i=1}^{m} s_i(t)e^{-\lambda t}\delta(x - x_i). \tag{21}$$

Let $v_j(x,\lambda)$ be a solution to the problem

$$L^* v_j^* + \lambda v_j^* = \delta(x - y_j), B^* v_j^*\big|_\Gamma = 0, \tag{22}$$

where L^* – formally adjoint operator to operator L, $B\ v = v$, if $Bu = u$ and $B^* v = \frac{\partial v}{\partial \nu} + (\sigma + (\vec{a}, \nu))v$ otherwise. The problem (22) is the adjoint problem to the problem

$$Lv + \lambda v = \delta(x - y_j), Bv|_\Gamma = 0. \tag{23}$$

Multiplying the equation (21) by v_j, integrating the result over Q, and using (20), we obtain the equalities

$$\left(v(T,x), v_j^*(T,x)\right) = \int_G v(T,x)v_j^*(T,x)dx = \sum_{i=1}^{m}\int_0^T s_i(t)e^{-\lambda t}dt v_j^*(x_i). \tag{24}$$

The equality (24) can be rewritten as

$$A(\lambda)\vec{S} = \vec{F}, \tag{25}$$

where the vectors $\vec{S}$, $\vec{F}$ have the coordinates $S_i = \int_0^T s_i(t)e^{-\lambda t}dt$ and $F_i = 4\pi\delta_j\left(v(T,x), v_j^*(x)\right)e^{\sqrt{\lambda}\delta_j}$ for $n = 3$ and $F_i = 2\sqrt{2\delta_j\pi}\lambda^{1/4}\left(v(T,x), v_j^*(T,x)\right)e^{\sqrt{\lambda}\delta_j}$ for $n = 2$. Transform the representation

$$f_j = \left(v(T,x), v_j^*(T,x)\right) = e^{-\lambda T}(w(T,x), (\lambda + L^*)^{-1}\delta(x - y_j)) =$$
$$e^{-\lambda T}((\lambda + L)^{-1}w(T,x), \delta(x - y_i)) = e^{-\lambda T}(\lambda + L)^{-1}w(T,x)\big|_{x=y_i},$$

Note that the last expression makes sense and these formal transformations are justified. Indicate that $w, w_t \in L_2\left(0,T; W_{p,B}^1(G)\right)$. In particular, we infer $w \in C\left([0,T]; W_p^1(G)\right)$ after a possible change on a set of zero measure. By embedding theorems, $w \in C\left([0,T]; L_q(G)\right)$ c $q \le 3p/(3 - p)$ for $n = 2,3$. In this case the expression $(\lambda + L)^{-1}w(T,x) \in W_q^2(G)$ is well-defined if the parameter λ is sufficiently large, say $\lambda \ge \lambda_0 > 0$ for some λ_0. However, $W_q^2(G) \subset C(\overline{G})$ when $n = 2,3$ and $q > 3/2$. Thus, we can consider the value $(\lambda + L)^{-1}w(T,x)\big|_{x=y_j}$. There is the estimate

$$|(\lambda+L)^{-1}w(T,y_j)|\le\|(\lambda+L)^{-1}w(T,x)\|_{C(\bar{G})}\le c_0\|(\lambda+L)^{-1}w(T,x)\|_{W_q^2(G)}\le c_1\|w(T,x)\|_{L_q(G)}, \quad (26)$$

where the constants do not depend on the parameter $\lambda\ge\lambda_0$ and we use resolvent estimates for the elliptic operators (see. [27, Ch.2]). As a consequence, we obtain the estimate

$$|F_j|\le c_2\lambda^{\gamma}e^{-T\lambda}e^{\sqrt{\lambda}\delta_j}\ \forall\lambda\ge\lambda_0,$$

where is the constant c_2 does not depend on λ and $\gamma=0$ when $n=3$ and $\gamma=1/4$ when $n=2$. Fix an arbitrary $\varepsilon\in(0,T)$. The above estimate implies that there exists a constant $C_0(\varepsilon)>0$ such that

$$|F_j|\le\frac{C_0(\varepsilon)e^{-(T-\varepsilon)\lambda}}{|\lambda|}\ \ \forall\lambda\ge\lambda_0. \quad (27)$$

By Theorem 1, the entries $b_{ij}(\lambda)$ of $A(\lambda)$ are representable as

$$b_{ji}(\lambda)=4\pi\delta_j v_j^*(x_i)e^{\sqrt{\lambda}\delta_j}=a_{ji}\left(1+O\left(\frac{1}{\sqrt{\lambda}}\right)\right)$$

for $n=3$ and

$$b_{ji}(\lambda)=2\sqrt{2\delta_j\pi}v_j^*(x_i)\lambda^{1/4}e^{\sqrt{\lambda}\delta_j}=a_{ji}\left(1+O\left(\frac{1}{\sqrt{\lambda}}\right)\right)$$

for $n=2$. Under the condition (13), we can assume that the matrix $A(\lambda)$ is invertible for $\lambda\ge\lambda_0$ and the elements of the inverse matrix $A^{-1}=\{s_{ij}\}$ are bounded by a constant independent of λ; otherwise, we increase the parameter λ_0. Therefore, we have

$$S_i(\lambda)=\sum_{j=1}^{m}s_{ij}(\lambda)F_j(\lambda)$$

and estimate (27) ensures that

$$|S_i(\lambda)|\le\frac{C_1(\varepsilon)(\varepsilon)e^{-(T-\varepsilon)\lambda}}{|\lambda|}\ \forall\lambda\ge\lambda_0. \quad (28)$$

Consider the functions $S_i(\lambda_0+z)$, where z is a complex parameter, $\operatorname{Re}z\ge0$. The function $S_i(\lambda_0+z)=\int_0^T s_i(t)e^{-\lambda_0 t}e^{-zt}dt$ is the Laplace transform of the function $\tilde{s}_i(t)=s_i(t)e^{-\lambda_0 t}$ for $t\le T$ and $\tilde{s}_i(t)=0$ for $t>T$. Introduce an additional function $W(z)=ze^{z(T-\varepsilon)}S_i(\lambda_0+z)$. It is analytic in the right half-plane and is bounded by a constant C_1 on the real semi-axis $\mathbb{R}^+$. Estimate this function on the on the imaginary axis. Integrating by parts, we have

$$S_i(\lambda_0+z)=-\frac{1}{\lambda_0+z}\left(s_i(T)e^{-\lambda_0 T}e^{-zT}+\int_0^T s_i{'}(t)e^{-\lambda_0 t}e^{-zt}dt\right).$$

For $z=iy$ we have the estimate

$$|W(z)|\le\left(|s_i(T)|+\|s_i{'}\|_{L_1(0,T)}\right)=C_3\ \forall z=iy,y\in R. \quad (29)$$

In each of the sectors $0\le\arg z\le\pi/2$, $-\pi/2\le\arg z\le0$ the function $W(z)$ admits the estimate

$$|W(z)|\le e^{|z|(T-\varepsilon)}\left(|s_i(T)|+\|s_i\|_{L_1(0,T)}\right)\ \forall\operatorname{Re}z\ge0. \quad (30)$$

Applying the Fragment–Lindelef Theorem (see theorem 5.6.1 in [28]) we obtain that in each of the sectors $0\le\arg z\le\pi/2$, $-\pi/2\le\arg z\le0$ the function $W(z)$ admits the estimate

$$W(z)\le\max(C_1,C_3)=C_4\ \ \forall\operatorname{Re}z\ge0. \quad (31)$$

Therefore,

$$|S_i(\lambda_0+z)|=|L(s_i(t))(z)|\le C_4(\varepsilon)e^{-(T-\varepsilon)\operatorname{Re} z}/|z|\ \forall \operatorname{Re} z\ge 0. \qquad (32)$$

We have equality ($\sigma\ge\lambda_0, p=\sigma+i\xi$)

$$s_i(t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}e^{pt}L(s_i)(p)\,dp=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{\sigma t}e^{i\xi t}L(s_i)(\sigma+i\xi)\,d\xi.$$

and, thereby,

$$s_i(t)e^{-\sigma(t-(T-\varepsilon))}=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\xi t}e^{\sigma(T-\varepsilon)}L(s_i)(\sigma+i\xi)\,d\xi.$$

The Parseval identity yields

$$\|s_i(t)e^{-\sigma(t-(T-\varepsilon))}\|^2_{L_2(-\infty,\infty)}=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{2\sigma(T-\varepsilon)}|L(s_i)(\sigma+i\xi)|^2\,d\xi\le\frac{C_5(\varepsilon)}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\sigma^2+\xi^2}\,d\xi\le\frac{C_5(\varepsilon)}{2\sigma}.$$

Since this inequality is true for all $\sigma>0$, $\tilde{s}_i(t)=0$ for $t\le T-\varepsilon$. Since the parameter ε is arbitrary, $s_i(t)=0$ for $t\le T$. We infer $N_i(t)=0$ for $t\le T$ and every i and, therefore, the right-hand side of (15) vanishes which implies that $u=0$. □

We note that the following condition is actually a necessary condition for the uniqueness of solutions to the problem (1)–(3). If it fails then any number of the points $\{y_i\}$ does not ensure uniqueness of solutions (see examples below).

Condition (A). For $n=2$, any three points $\{y_i\}$ do not lie on the same straight line and, for $n=3$ any four points $\{y_i\}$ do not lie on the same plane.

Next, we describe some model situation in which $Lu=-\Delta u+\lambda_0 u$, $\lambda_0\ge 0$, $G=\mathbb{R}^n$ and functions N_i on the right-hand side of (1) are real constants.

Theorem 4. Let u_1,u_2 be two solutions to the problem (1)–(3) from class described in the theorem 1 with the right-hand sides in (1) of the form $\sum_{i=1}^{r_j}N_i^j\delta(x-x_i)$ $(N_j=\text{const}, j=1,2)$, the condition (A) holds, and $s\ge 2r+1$ in the case $n=2$ and $s\ge 3r+1$ in the case $n=3$, where $r\ge\max(r_1,r_2)$ (i. e., there is the upper bound for the number $\max(r_1,r_2)$). Then $u_1=u_2$, $r_1=r_2$, and $N_i^1=N_i^2$ for all i, i. e., a solution to the problem of recovering the number m, points x_i, and constants N_i is unique.

Proof. Let the functions u_1,u_2 do not coincide and let $w=u_1-u_2$. The function w satisfies the homogeneous initial data and over determination conditions (3) and we have (after renumbering the constants N_i^j and points x_i)

$$w_t+Lw=\sum_{i=1}^{r_3}N_i\delta(x-x_i)-\sum_{i=1}^{r_4}C_i\delta(x-x_i^*),\ 2r\ge r_3+r_4, N_i, C_i=\text{const}, \qquad (33)$$

where $N_i,C_j>0$ for all i,j. Without loss of generality, we can assume that all the numbers N_i,C_i are not equal to zero and all points x_i,x_i are distinct. Let, for example, $n=3$. For simplicity, take $\lambda_0=0$. The proof is the same for other values of this parameter. Applying the Laplace transform, we infer

$$\hat{w}(x)=\sum_{i=1}^{r_3}\frac{N_i}{4\pi|x-x_i|\lambda}e^{-\sqrt{\lambda}|x-x_i|}-\sum_{i=1}^{r_4}\frac{C_i}{4\pi|x-x_i^*|\lambda}e^{-\sqrt{\lambda}|x-x_i^*|}. \qquad (34)$$

Using (3), we obtain

$$\sum_{i=1}^{r_3}\frac{N_i}{4\pi|y_j-x_i|\lambda}e^{-\sqrt{\lambda}|y_j-x_i|}=\sum_{i=1}^{r_4}\frac{C_i}{4\pi|y_j-x_i^*|\lambda}e^{-\sqrt{\lambda}|y_j-x_i^*|},\ j=1,2,\ldots,s. \qquad (35)$$

For definiteness, we assume that $r_3 \geq r_4$. Let us show that the sets of numbers $\{r_{ij} = | x_i - y_j | : i = 1, 2 \ldots, r_3\}$, $\{r_{ij} = | x_i^* - y_j |, i = 1, 2 \ldots, r_4\}$ coincide for all j. Fix the parameter j. Let $\delta_{1j} = \min_i r_{ij}$, $\delta_{1j} = \min_i r_{ij}$. Demonstrate that $\delta_{1j} = \delta_{1j}$. Assume the contrary. Let, for example, $\delta_{1j} < \delta_{1j}$. Multiply the system (35) by $4\pi\delta_{1j}\lambda e^{\sqrt{\lambda}\delta_{1j}}$ and passing to the limit as $\lambda \to +\infty$ we obtain the equality

$$\sum_{i:|x_i - y_j| = \delta_{1j}} N_i = 0. \tag{36}$$

It is a contradiction, since $N_i > 0$. So, $\delta_{1j} = \delta_{1j}$ and multiplying the system (35) by $4\pi\delta_{1j}\lambda e^{\sqrt{\lambda}\delta_{1j}}$ and passing to the limit as $\lambda \to +\infty$ we also derive that

$$\sum_{i:|x_i - y_1| = \delta_{1j}} N_i = \sum_{i:|x_i^* - y_j| = \delta_{1j}} C_i.$$

So, we can reduce the following sums on the left and on the right in the equalities (35):

$$\sum_{i:|x_i - y_1| = \delta_{1j}} \frac{N_i}{4\pi | y_j - x_i | \lambda} e^{-\sqrt{\lambda}|y_j - x_i|}, \quad \sum_{i:|x_i^* - y_j| = \tilde{\delta}_{1j}} \frac{C_i}{4\pi | y_j - x_i^* | \lambda} e^{-\sqrt{\lambda}|y_j - x_i^*|}.$$

Denote $\delta_{2j} = \min_{i:r_{ij} > \delta_{1j}} r_{ij}$ and $\delta_{2j} = \min_{i:r_{ij} > \delta_{1j}} r_{ij}$. Repeating the arguments, we obtain that $\delta_{2j} = \delta_{2j}$ and, thereby,

$$\sum_{i:|x_i - y_j| = \delta_{2j}} N_i = \sum_{i:|x_i^* - y_j| = \tilde{\delta}_{2j}} C_i.$$

Again, abbreviated equal summands (35), we arrive at the system (35), where the sums on the left and on the right are taken over $i : r_{ij} > \delta_{2j}$ и $i : r_{ij} > \delta_{2j}$, respectively. It is now obvious by induction that there are pairs of equal numbers δ_{kj}, δ_{kj} $k = 1, 2, \ldots, r_{0j} \leq \min(r_3, r_4)$ and

$$\sum_{i:|x_i - y_j| = \delta_{kj}} N_i = \sum_{i:|x_i^* - y_j| = \tilde{\delta}_{kj}} C_i, k = 1, 2, \ldots, r_{0j}, \tag{37}$$

moreover, the left-hand and right-hand sides of these equalities are positive. So, the sets of numbers $\{r_{ij} = | x_i - y_j | : i = 1, 2 \ldots, r_3\}$, $\{\tilde{r}_{ij} = | x_i^* - y_j |, i = 1, 2 \ldots, r_4\}$ coincide for all j. In particular, it follows that for any point, for example, x_1 and any j, there exists a point x_{i_j} such that

$$| x_1 - y_j | = | x_{i_j}^* - y_j |, \ j = 1, 2, \ldots, s.$$

But we have $s \geq 3r + 1$ and $r_4 \leq r$ is the number of points $\{x_i^*\}$. Hence, among the points $\{x_{i_j}^*\}_{j=1}^{s}$ there are four coinciding points. After renumbering if necessary we can assume that these points are $x_{i_1}^*, x_{i_2}^*, x_{i_3}^*, x_{i_4}^*$. Then the equalities

$$| x_1 - y_j | = | x_{i_j}^* - y_j |, \ j = 1, 2, \ldots, s.$$

imply that the points y_j with $j = 1, 2, 3, 4$ lie in the same plane which is perpendicular to the segment $\left[x_1, x_{i_1}\right]$, but this fact contradicts to the conditions (A). So, $w \equiv 0$.

The proof in the case of $n = 2$ is almost the same but we use an asymptotic representation for a fundamental solution $\frac{i}{4} H_0^{(1)}(i\sqrt{\lambda} | x - x_0 |)$ defined by the equality (7), where $\psi \equiv 0$. As in the case of $n = 3$, we arrive at contradiction with the condition (A). □

We display the corresponding examples showing the accuracy the results obtained. The following example shows that if the condition (A) fails then the problem of recovering the intensities of sources (sinks) located at x_1, x_2 has a nonunique solution. At the same time, it is an example of the nonuniqueness in the problem of recovering the intensity of one source and its location. Note that the problem of determining the location of one source x_0 and its intensity $N(t)$ is simple enough and to uniquely recover these parameters we need two measurements in the case of $n=1$ [22], three measurements in the case of $n=2$ [28] and four measurements (that is $s=4$ in (3)) in the case of $n=4$ [25]. The smaller number of points does not allow to define the parameters $N(t), x_0$ uniquely. We should also require that the point x_0 lie between two measurement points in the case of $n=1$ and the condition (A) holds in the case of $n=2,3$. The numerical solution of the problem of recovering one source is treated in the articles [6, 9–15, 19, 28].

Example 1. First we take $n=3$, $G=\mathbb{R}^n$, $Lu=-\Delta u$. Let u be a solution to the equation (1) satisfying the homogeneous initial conditions with the right-hand side in (1) of the form

$$N(t)\big(\delta(x-x_1)-\delta(x-x_2)\big).$$

The Laplace transform of this solution to the problem (1)–(2) is written as

$$\hat{u}=\hat{N}(\lambda)\Big(\frac{1}{4\pi|x-x_1|}e^{-\sqrt{\lambda}|x-x_1|}-\frac{1}{4\pi|x-x_2|}e^{-\sqrt{\lambda}|x-x_2|}\Big).$$

Let P be the plane perpendicular to the segment $[x_1, x_2]$ and passing through its center. We have

$$\hat{u}(y,\lambda)\equiv 0 \ \ \forall y\in P.$$

So, $u(y,t)=0$ for all $y\in P$. Precisely the same example can be constructed in the case $n=2$. We take the perpendicular to the segment $[x_1,x_2]$ passing through its center rather than the plane P. Thus, if condition (A) fails then any number of measurement points does not allow to determine the intensity and the location of the sources.

Example 2. Consider the case of $G=\mathbb{R}^n$, $Lu=-\Delta u$. Let us show that the conditions (3) with $s=4$ in the case of $n=2$ and $s=6$ in the case of $n=3$ does not allow to determine location of two sources and their intensities even if the condition (A) holds. Let u_1, u_2 be solutions to the equation (10) satisfying the homogeneous initial conditions in which the right-hand sides are of the form

$$N(t)\delta(x-x_1)+N(t)\delta(x-x_2), N(t)\delta(x-x_1^*)+N(t)\delta(x-x_2^*).$$

Let, for example, $n=3$. Then the Laplace transforms of u_1,u_2 are as follows:

$$\hat{u}_1(x,\lambda)=\sum_{i=1}^{2}\frac{\hat{N}}{4\pi|x-x_i|}e^{-\sqrt{\lambda}|x-x_i|}, \hat{u}_2(x,\lambda)=\sum_{i=1}^{2}\frac{\hat{N}}{4\pi|x-x_i^*|}e^{-\sqrt{\lambda}|x-x_i^*|}. \tag{38}$$

Here we use explicit representations of the fundamental solution for the Helmgoltz equation (see, for example, in [30, §3.1] or [31, ch. 4, 8]). We take $x_1=(a,a,0), x_1^*=(a,-a,0)$, $x_2=(-a,-a,0), x_2^*=(-a,a,0)$ ($a>0$). As is easily seen, the functions $\hat{u}_1,\hat{u}_2$ coincide at the points $y_1=(M,0,0), y_2=(-M,0,0), y_3=(0,M,0), y_4=(0,-M,0), y_5=(0,0,M), y_6=(0,0,-M)$, where $M>0$ and, thus, the problem of recovering the locations of 2 sources and their intensities admits several solutions in the case of $s=6$. It follows from the theorem 2 that in the case of $s=7$ points x_1,x_2 and the intensities are determined uniquely (if the condition (A) holds and the intensities are constants).

Consider the case of $n=2$. As before, we construct functions u_1,u_2 whose Laplace transform is of the form

$$\hat{u}_1=\sum_{j=1}^{2}\frac{iN}{4\lambda}H_0^{(1)}(i\sqrt{\lambda}\,|x-x_j|), \hat{u}_2=\sum_{j=1}^{2}\frac{i\hat{N}}{4\lambda}H_0^{(1)}(i\sqrt{\lambda}\,|x-x_j^*|),$$

where H_0^1 is the Hankel functions [32]. Let us take $x_1=(a,a), x_1^*=(a,-a), x_2=(-a,-a), x_2^*=(-a,a)$ ($a>0$). It is easy to check that

$$\hat{u}_1(y_j,\lambda)=\hat{u}_2(y_j,\lambda)\ \forall j=1,\ldots,4,\lambda\in\mathbb{R}^{+}. \tag{39}$$

where $y_1=(M,0), y_2=(-M,0), y_3=(0,M), y_4=(0,-M)$. It follows from the theorem 2 that the points x_1,x_2 and intensities are determined uniquely in case $s=5$ (if condition (A) holds and the intensities are constants).

Remark 1. The examples show that the number of minima of the corresponding objective functionals introduced if we solve the problem (1)–(3) numerically reducing the problem to an optimal control problem can be large and even can be a manifold.

Remark 2. Relying on asymptotic representations and Theorem 1 in the case of constant values N_i, we can construct a numerical algorithm for finding sources $\{x_i\}$ employing the ideas from the article [19]. Some review of the results connected with numerical determining point sources can be found in the article [33] and some results in [34–37].

References

1. Marchuk G.I. *Mathematical Models in Environmental Problems. Studies in Mathematics and its Applications*. Elsevier Science Publishers, Amsterdam, 1986, Vol. 16, 216 p.

2. Ozisik M.N., Orlande H.R.B. *Inverse Heat Transfer: Fundamentals and Applications.* Taylor & Francis, New York, 2000, 330 p.

3. Alifanov O.M., Artyukhin E.A., Nenarokomov A.V. *Obratnye zadachi v issledovanii slozhnogo teploobmena* (Inverse problems of complex heat transfer). Moscow, Yanus-K Publ., 2009, 299 p. (in Russ.).

4. Panasenko E.A., Starchenko A.V. Numerical Solution of Some Inverse Problems with Various Types of Sources of Atmospheric Pollution. *Vestn. Tomsk. Gos. Univ. Mat. Mekh.*, 2008, no. 2(3), pp. 47–55. (in Russ.).

5. Penenko V.V. Variational methods of data assimilation and inverse problems for studying the atmosphere, ocean, and environment. *Numerical Analysis and Applications*, 2009, Vol. 2, Iss. 4, pp. 341–351.

6. Yang, C.Y. Solving the two-dimensional inverse heat source problem through the linear least-squares error method. *International Journal of Heat and Mass Transfer*, 1998, Vol. 41, no. 2, pp. 393–398. DOI: 10.1016/S0017-9310(97)00125-7

7. Starchenko A.V., Panasenko E.A. Parallel Algorithms for the Decision of Return Problems of Carrying over of the Impurity. *Vestnik USATU*, 2010, Vol. 14, no. 5(40), pp. 133–139. (in Russ.).

8. Mamonov A. V., Tsai Y-H. R. Point source identification in nonlinear advection-diffusion-reaction systems. *Inverse Problems,* 2013, Vol. 29, no. 3, p. 26. DOI: 10.1088/0266-5611/29/3/035009

9. Deng X., Zhao Y., Zou J. On linear finite elements for simultaneously recovering source location and intensity. *Int. J. Numer. Anal. Model*, 2013, Vol. 10, no. 3, pp. 588–602.

10. Verdiere N., Joly-Blanchard G., Denis-Vidal L. Identifiability and Identification of a Pollution Source in a River by Using a Semi-Discretized Model. *Applied Mathematics and Computation*, 2013, vol. 221, pp. 1–9. DOI: 10.1016/j.amc.2013.06.022

11. Mazaheri M., Samani J.M.V., Samani H.M.V. Mathematical Model for Pollution Source Identification in Rivers. *Environmental Forensics*, 2015, Vol. 16, Iss. 4, pp. 310–321. DOI: 10.1080/15275922.2015.1059391

12. Su J. Heat Source Estimation with the Conjugate Gradient Method in Inverse Linear Diffusive Problems. *J. Braz. Soc. Mech. Sci.*, 2001, Vol. 23, no. 3, pp. 321–334. DOI: 10.1590/s0100-73862001000300005

13. Neto A.J.S., Oziik M.N. Twodimensional inverse heat conduction problem of estimating the timevarying strength of a line heat source. *Journal of Applied Physics,* 1992, Vol. 71, Iss. 11, pp. 53–57. DOI: 10.1063/1.350554

14. Milnes E., Perrochet P. Simultaneous Identification of a Single Pollution Point-Source Location and Contamination Time under Known Flow Field Conditions. *Advances in Water Resources*, 2007, Vol. 30, iss. 12, pp. 2439–2446. DOI: 10.1016/j.advwatres.2007.05.013

15. Liu F.B. A modified genetic algorithm for solving the inverse heat transfer problem of estimating plan heat source. *International Journal of Heat and Mass Transfer,* 2008, Vol. 51, Iss. 15-16, pp. 3745–3752. DOI: 10.1016/j.ijheatmasstransfer.2008.01.002

16. Penenko A.V., Rachmetullina S. Algorithms for Atmospheric Emission Source Localization Based on the Automated Ecological Monitoring System Data. *Siberian Electronic Mathematical Reports. Proceedings of conferences*, 2013, Vol. 10, pp. S35–S54. (in Russ.).

17. Badia A. El, Ha-Duong T., Hamdi A. Identification of a Point Source in a Linear Advection-Dispersion-Reaction Equation: Application to a Pollution Source Problem. *Inverse Problems*, 2005, Vol. 21, no. 3, pp. 1121–1136. DOI: 10.1088/0266-5611/21/3/020

18. Badia A. El, Hamdi A. Inverse Source Problem in an Advection-Dispersion-Reaction System: Application to Water Pollution. *Inverse Problems*, 2007, Vol. 23, no. 5, pp. 2103–2120. DOI: 10.1088/0266-5611/23/5/017

19. Badia A. El, Ha-Duong T. Inverse Source Problem for the Heat Equation: Application to a Pollution Detection Problem. *J. Inverse Ill-Posed Probl.*, 2002, Vol. 10, Iss. 6, pp. 585–599. DOI: 10.1515/jiip.2002.10.6.585

20. Badia A. El, Ha-Duong T. An inverse Source Problem in Potential Analysis. *Inverse Problems*, 2000, Vol. 16, Iss. 3, pp. 651–663. DOI: 10.1088/0266-5611/16/3/308

21. Ling L., Takeuchi T. Point Sources Identification Problems for Heat Equations. *Commun. Comput. Phys.*, 2009, Vol. 5, no. 5, pp. 897–913.

22. Pyatkov S.G., Safonov E.I. Point Sources Recovering Problems for the One-Dimensional Heat Equation. *Journal of Advanced Research in Dynamical and Control Systems*, 2019, Vol. 11, Iss. 01, pp. 496–510. http://www.jardcs.org/abstract.php?id=100#

23. Pyatkov S.G., Neustroeva L.V. On Some Asymptotic Representations of Solutions to Elliptic Equations and Their Applications. *Complex Variables and Elliptic Equations*, 2021, Vol. 66, no. 6-7, pp. 964–987. DOI: 10.1080/17476933.2020.1801656

24. Triebel H. *Interpolation Theory. Function Spaces. Differential Operators*. Berlin: VEB Deutscher Verlag der Wissenschaften, 1978, 528 p.

25. Amann H. Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems. Function Spaces, *Differential Operators and Nonlinear Analysis*. Teubner-Texte Math. Bd. 133. Stuttgart: Teubner, 1993, pp. 9–126. DOI: 10.1007/978-3-663-11336-2_1

26. Neustroeva L.V., Pyatkov S.G. On recovering a point source in some heat and mass transfer problems. *AIP Conference Proceedings*, 2021, Vol. 2328, p. 020006. DOI: 10.1063/5.0042357

27. Denk R., Hieber R.M., Prüss J. R-boundedness, Fourier multipliers and problems of elliptic and parabolic type. *Mem. Amer. Math. Soc.*, 2003, Vol. 166, no. 788. DOI: 10.1090/memo/0788

28. Hsu Y-K., Holsen T.M., Hopke P.K. Comparison of hybrid receptor models to locate PCB sources in Chicago. *Atmospheric Environment*, 2003, Vol. 37, pp. 545–562.

29. Safonov E., Pyatkov S. On Some Classes of Inverse Problems on Determining the Source Function. *Proc. of the 8th Scientific Conference on Information Technologies for Intelligent Decision Making Support (ITIDS 2020)*. pp. 242–248. DOI: 10.2991/aisr.k.201029.047

30. Vladimirov V.S., Zharinov V.V. *Uravneniya matematicheskoy fiziki: ucheb. dlya studentov vuzov* (Equations of Mathematical Physics: Textbook for University Students). Moscow Fizmatlit Publ., 2004, 398 p. (in Russ.).

31. Sveshnikov A.G., Bogolyubov A.N., Kravtsov V.V. Lektsii po matematicheskoy fizike (Lectures on Mathematical Physics). Moscow, MGU Publ., 1993, 351 p. (in Russ.).

32. Watson G.N. A Treatise on the Theory of Bessel Functions. Cambridge: Cambridge University Press, 1944, 804 p.

33. Kozhevnikova M.F., Levenets V.V., Rolik I.L. The Pollution Source Identification: Computational Approach. *Problems of Atomic Science and Technology*, 2011, no. 6, pp. 149–156.

34. Zhou L., Hopke P.K., Liu W. Comparison of Two Trajectory Based Models for Locating Particle Sources for Two Rural New York Sites. *Atmospheric Environment*, 2004, Vol. 38, Iss. 13, pp. 1955–1963. DOI: 10.1016/j.atmosenv.2003.12.034

35. Han Y.-J., Holsen T.M., Hopke P.K., Cheong J.-P., Kim H., Yi S.-M. Identification of Source Location for Atmospheric Dry Deposition of Heavy Metals During Yellow-Sand Events in Seoul, Korea in 1998 Using Hybrid Receptor Models. *Atmospheric Environment*, 2004, Vol. 38, pp. 5353–5361. DOI: 10.1016/j.atmosenv.2004.02.069

36. Pekney N.J., Davidson C.I., Zhow L., Hopke P.K. Application of PSCF and CPF to PMF-Modeled Sources of $PM_{2.5}$ in Pittsburgh. *Aerosol Science and Technology*, 2006, Vol. 40, Iss. 10, pp. 952–961. DOI: 10.1080/02786820500543324

37. Tichmarsh E.C. *Theory of functions*. Oxford, Oxford University press, 1939, 454 p.

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О ЕДИНСТВЕННОСТИ В ЗАДАЧАХ ОПРЕДЕЛЕНИЯ ТОЧЕЧНЫХ ИСТОЧНИКОВ В МАТЕМАТИЧЕСКИХ МОДЕЛЯХ ТЕПЛОМАССОПЕРЕНОСА

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Аннотация. В работе рассмотрены задачи об определении точечных источников для математических моделей тепломассопереноса. В качестве условий переопределения берутся значения решения (концентраций) в некоторых точках лежащих внутри области. Рассматривается параболическое уравнение второго порядка, в правой части которого присутствует линейная комбинация дельта-функций Дирака $\delta(x–x_i)$ с коэффициентами, зависящими от времени и характеризующими мощность источников. Рассматриваются несколько различных задач, в том числе задача определения интенсивностей источников в случае, если их местоположение задано.

В этом случае мы приводим теорему единственности решений, доказательство которой основано на теореме Фрагмена–Линделефа. Далее в модельном случае мы рассматриваем задачу об одновременном определении мощностей источников и их местоположения. Описаны условия на числе замеров (условий переопределения), когда решение определяется единственным образом. Приведены примеры, показывающие точность полученных результатов. Проблема возникает при решении экологических задач, прежде всего задач определения источников загрязнения в водоеме или атмосфере. Результаты важны при построении численных алгоритмов решения задачи. В литературе такие задачи решаются численно с помощью сведения задачи к задаче оптимального управления и минимизации соответствующего целевого функционала. Примеры показывают, что такой способ решения не всегда корректен, поскольку целевой функционал может иметь значительное количество минимумов.

Ключевые слова: тепломассоперенос; параболическое уравнение; единственность; обратная задача; точечный источник.

Литература

1. Marchuk, G.I. Mathematical Models in Environmental Problems / G.I. Marchuk. – Elsevier Science Publishers, Amsterdam, 1986. – 216 p.

2. Ozisik M.N., Orlande H.R.B. Inverse Heat Transfer: Fundamentals and Applications / M.N. Ozisik, H.R.B. Orlande. – Taylor & Francis, New York, 2000. – 330 p.

3. Алифанов, О.М. Обратные задачи в исследовании сложного теплообмена / О.М. Алифанов, Е.А. Артюхин, А.В. Ненарокомов. – Москва: Янус-К, 2009. – 299 с.

4. Панасенко, Е.А. Численное решение некоторых обратных задач с различными типами источников атмосферного загрязнения / Е.А. Панасенко, А.В. Старченко // Вестн. Томск. гос. ун-та. Матем. и мех. – 2008. – № 2(3). – С. 47–55.

5. Пененко, В.В. Вариационные методы усвоения данных и обратные задачи для изучения атмосферы, океана и окружающей среды / В.В. Пененко // Сиб. журн. вычисл. Матем. – 2009. – Т. 12, № 4. – С. 421–434.

6. Yang, C.Y. Solving the two-dimensional inverse heat source problem through the linear least-squares error method / C.Y. Yang // International Journal of Heat and Mass Transfer. – 1998. – Vol. 41, no. 2. – P. 393–398.

7. Старченко, А.В. Параллельные алгоритмы для решения обратных задач переноса примеси / А.В. Старченко, Е.А. Панасенко // Вестник Уфимского государственного авиационного технического университета. – 2010. – Т. 14, № 5(40). – С. 133–139.

8. Mamonov, A.V. Point Source Identification in Nonlinear Advection-Diffusion-Reaction Systems / A.V. Mamonov, Y-H.R. Tsai // Inverse Problems. – 2013. – Vol. 29, no. 3. – p. 26.

9. Deng, X. On Linear Finite Elements for Simultaneously Recovering Source Location and Intensity / X. Deng, Y. Zhao, J. Zou // Int. J. Numer. Anal. Model. – 2013. – Vol. 10, no. 3. – P. 588–602.

10. Verdiere, N. Identifiability and Identification of a Pollution Source in a River by Using a Semi-Discretized Model / N. Verdiere, G. Joly-Blanchard, L. Denis-Vidal // Applied Mathematics and Computation. – 2013. – Vol. 221. – P. 1–9.

11. Mazaheri, M. Mathematical Model for Pollution Source Identification in Rivers / M. Mazaheri, J.M.V. Samani, H.M.V. Samani // Environmental Forensics. – 2015. – Vol. 16, Iss. 4. – P. 310–321.

12. Su, J. Heat Source Estimation with the Conjugate Gradient Method in Inverse Linear Diffusive Problems / J. Su // J. Braz. Soc. Mech. Sci. – 2001. – Vol. 23, no. 3. – P. 321–334.

13. Neto, A.J.S. Twodimensional Inverse Heat Conduction Problem of Estimating the Timevarying Strength of a Line Heat Source / A.J.S. Neto, M.N. Oziik// Journal of Applied Physics. – 1992. – Vol. 71, Iss. 11. – P. 53–57.

14. Milnes, E. Simultaneous Identification of a Single Pollution Point-Source Location and Contamination Time under Known Flow Field Conditions / E. Milnes, P. Perrochet // Advances in Water Resources. – 2007. – Vol. 30, Iss. 12. – P. 2439–2446.

15. Liu, F.B. A Modified Genetic Algorithm for Solving the Inverse Heat Transfer Problem of Estimating Plan Heat Source / F.B. Liu // International Journal of Heat and Mass Transfer. – 2008. – Vol. 51, Iss. 15-16. – P. 3745–3752.

16. Пененко А.В., Рахметуллина С.Ж. Алгоритмы локализации источников загрязнения атмосферного воздуха на основе данных автоматизированной системы экологического мониторинга. Сиб. электрон. матем. Изв. – 2013. – Vol. 10. – С.35–С.54

17. Badia, A.El Identification of a Point Source in a Linear Advection-Dispersion-Reaction Equation: Application to a Pollution Source Problem / A. El Badia, T. Ha-Duong, A. Hamdi // Inverse Problems. – 2005. – Vol. 21, no. 3. – P. 1121–1136.

18. Badia A.El Inverse Source Problem in an Advection-Dispersion-Reaction System: Application to Water Pollution / A. El Badia, A. Hamdi // Inverse Problems. – 2007. – Vol. 23, no. 5. – P. 2103–2120.

19. Badia A.El Inverse Source Problem for the Heat Equation: Application to a Pollution Detection problem / A. El Badia, T. Ha-Duong // J. Inverse Ill-Posed Probl. – 2002. – Vol. 10, Iss. 6. – P. 585–599.

20. Badia, A.El An Inverse Source Problem in Potential Analysis / A. El Badia, T. Ha-Duong // Inverse Problems. – 2000. – Vol. 16, Iss. 3. – P. 651–663.

21. Ling, L. Point Sources Identification Problems for Heat Equations / L. Ling, T. Takeuchi // Commun. Comput. Phys. – 2009. – Vol. 5, no. 5. – P. 897–913.

22. Pyatkov, S.G. Point Sources Recovering Problems for the One-Dimensional Heat Equation / S.G. Pyatkov, E.I. Safonov // Journal of Advanced Research in Dynamical and Control Systems. – 2019. – Vol. 11, Iss. 01. – P. 496–510. http://www.jardcs.org/abstract.php?id=100#

23. Pyatkov, S.G. On Some Asymptotic Representations of Solutions to Elliptic Equations and Their Applications / S.G. Pyatkov, L.V. Neustroeva // Complex Variables and Elliptic Equations. – 2021. – Vol. 66, no. 6-7. – P. 964–987.

24. Triebel, H. Interpolation Theory. Function Spaces. Differential Operators / H. Triebel // Berlin: VEB Deutscher Verlag der Wissenschaften, 1978. – 528 p.

25. Amann, H. Nonhomogeneous Linear and Quasilinear Elliptic and Parabolic Boundary Value Problems / H. Amann // Function Spaces, Differential Operators and Nonlinear Analysis. Teubner-Texte Math. Bd. 133. – Stuttgart: Teubner, 1993. – P. 9–126.

26. Neustroeva L.V., Pyatkov S.G. On Recovering a Point Source in Some Heat and Mass Transfer Problems / L.V. Neustroeva, S.G. Pyatkov // AIP Conference Proceedings. – 2021. – Vol. 2328. – P. 020006.

27. R-boundedness, Fourier Multipliers and Problems of Elliptic and Parabolic Type / R. Denk, R.M. Hieber, J. Prüss // Mem. Amer. Math. Soc. – 2003. – Vol. 166, no. 788.

28. Comparison of Hybrid Receptor Models to Locate PCB Sources in Chicago / Y-K. Hsu, T.M. Holsen, P.K. Hopke // Atmospheric Environment. – 2003. – Vol. 37. – P. 545–562.

29. Safonov, E. On Some Classes of Inverse Problems on Determining the Source Function / E. Safonov, S. Pyatkov // Proc. of the 8th Scientific Conference on Information Technologies for Intelligent Decision Making Support (ITIDS 2020). –Atlantis Press, 2020. – P. 242–248.

30. Владимиров, В.С. Уравнения математической физики: учеб. для студентов вузов / В.С. Владимиров, В.В. Жаринов. – М.: Физматлит, 2004. – 398 с.

31. Свешников, А.Г. Лекции по математической физике / А.Г. Свешников, А.Н. Боголюбов, В.В. Кравцов. – М.: Изд-во МГУ, 1993. – 351 с.

32. Watson, G.N. A treatise on the theory of Bessel functions / G.N. Watson. – Cambridge: Cambridge University Press, 1944. – 804 p.

33. Kozhevnikova M.F., Levenets V.V., Rolik I.L. The Pollution Source Identification: Computational Approach / M.F. Kozhevnikova, V.V. Levenets, I.L. Rolik // Problems of Atomic Science and Technology. – 2011. – no. 6. – P. 149–156.

34. Zhou, L. Comparison of Two Trajectory Based Models for Locating Particle Sources for Two Rural New York sites / L. Zhou, P.K. Hopke, W. Liu // Atmospheric Environment. – 2004. – Vol. 38, Iss. 13. – P. 1955–1963.

35. Identification of Source Location for Atmospheric Dry Deposition of Heavy Metals During Yellow-Sand Events in Seoul, Korea in 1998 Using Hybrid Receptor Models / Y.-J. Han, T.M. Holsen, P.K. Hopke *et al.* // Atmospheric Environment. – 2004. – Vol. 38. – P. 5353–5361.

36. Application of PSCF and CPF to PMF-Modeled Sources of $PM_{2.5}$ in Pittsburgh / N.J. Pekney, C.I. Davidson, L. Zhow, P.K. Hopke // Aerosol Science and Technology. – 2006. – Vol. 40, Iss. 10. – P. 952–961.

37. Tichmarsh, E.C. Theory of Functions / E.C. Tichmarsh. – Oxford, Oxford University press, 1939. – 454 p.

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