DOI: 10.14529/mmph220305

ANALYSIS OF THE CLASS OF HYDRODYNAMIC SYSTEMS

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Abstract. The solvability of the Cauchy–Dirichlet problem for the generalized homogeneous model of the dynamics of the high-order viscoelastic incompressible Kelvin–Voigt fluid is considered. In the study, the theory of semilinear equations of the Sobolev type was used. The indicated problem for the system of differential equations in partial derivatives is reduced to the Cauchy problem for the indicated type of the equation. The theorem on the existence of the unique solution of this problem, which is a quasi-stationary trajectory, is proved, and its phase space is described.

Keywords: Sobolev type equation; phase space; viscoelastic incompressible fluid.

1. Formulation of the problem

System of equations

$$\begin{cases} (1-\text{æ}\nabla^2)v_t = \nu\nabla^2 v - (v\cdot\nabla)v + \sum\limits_{m=1}^{r}\sum\limits_{s=0}^{n_m-1} A_{m,s}\nabla^2 w_{m,s} - \nabla p, \\ 0 = \nabla\cdot v, \\ \dfrac{\partial w_{m,0}}{\partial t} = v + \alpha_m w_{m,0}, \qquad m=\overline{1,r}, \\ \dfrac{\partial w_{m,s}}{\partial t} = s w_{m,s-1} + \alpha_m w_{m,s}, \quad s=\overline{1,n_m-1}, \quad \alpha_m \in \mathbb{R}_-, \quad A_{m,s}\in\mathbb{R}_+, \end{cases} \tag{1}$$

describes the homogeneous generalized model of the dynamics of the high-order viscoelastic incompressible Kelvin–Voigt fluid [1].

The functions $v=(v_1,\ldots,v_n), v_i = v_i(x,t), \quad x\in\Omega$ have the physical meaning of the fluid flow velocity, the function $p = p(x,t)$ describes the pressure. Here, $\Omega\subset R^n, \quad n=2,3,4$ is a bounded domain with boundary $\partial\Omega$ of the class C^∞. The parameters $\nu\in R_+$ and $\text{æ}\in R$ correspond to the viscous and elastic properties of the liquid. The parameters $A_{m,s}$ define the pressure retardation time.

For the system (1), the Cauchy–Dirichlet problem is considered

$$\begin{cases} v(x,0)=v_0(x),\ p(x,0)=p_0(x),\ w_{m,s}(x,0)=w_{m,s}(x)\ \ \forall x\in\Omega, \\ v(x,t)=0,\ w_{m,s}(x,t)=0\ \ \forall(x,t)\in\partial\Omega\times R, \quad m=\overline{1,r},\ s=\overline{0,n_m-1}. \end{cases} \tag{2}$$

Разрешимость задачи (1), (2) рассматривается в рамках теории полулинейных уравнений соболевского типа [2, 3].

The solvability of problem (1), (2) is considered within the framework of the theory of semilinear Sobolev type equations [2, 3].

2. The solution of the problem

The proof of the existence theorem for the unique solution of the problem is that the Cauchy problem for the semi-linear Sobolev type equation [4] is first studied, and then the original problem is reduced to it.

Consider the Cauchy problem

$$u(0)=u_0 \tag{3}$$

for the semi-linear Sobolev type equation

$$\mathrm{L}\dot{u} = \mathrm{M}(u). \tag{4}$$

Here U and F are Banach spaces, operators $\mathrm{L} \in L(U;F)$ and $\mathrm{M} \in C^{\infty}(U;F)$.

Definition 1. *The solution of the problem (3), (4) is the vector function*

$$u \in C^{\infty}((-t_0;t_0);U),\ t_0 = t_0(u_0) > 0,$$

satisfying equation (4) and condition (3).

Problem (3), (4) is solvable not for all initial data from the Banach space U, and if the solution of this problem exists, then it may be not unique.

Definition 2. *A Banach* C^k *– manifold* B *is called the phase space of equation (4), if* $\forall u_0 \in B$ *there is a unique solution* $u = u(t)$ *of the problem (3), (4) on the interval* $(-t_0, t_0)$.

Definition 3. *The solution* $u = u(t)$ *of the problem (3), (4), for which* $\mathrm{L}\dot{u}^0 \equiv 0\ \forall t \in (-t_0;t_0)$, *where* $u^0 = \mathrm{P}u$, *is called a quasi-stationary trajectory of the equation (4).*

Here $u = u^0 + u^1,\ u^0 \in U^0,\ u^1 \in U^1,\ U = U^0 \oplus U^1$. P is the projection of the Banach space U onto U^0.

Let the operator L be bi-splitting, its kernel $\ker \mathrm{L}$ and image $\mathrm{im}\, \mathrm{L}$ be complemented in the spaces U and F respectively. Denote by $\mathrm{M}'_{u_0} \in L(U;F)$ the Fréchet derivative of the operator M at the point $u_0 \in U$ and introduce into consideration the chains $\mathrm{M}'_{u_0}-$ associated vectors of the operator L, which we will choose from some complement $\mathrm{coim}\, \mathrm{L}$ to the kernel $\ker \mathrm{L}$ in the Banach space U. Consider the condition

(C1). Regardless of the choice $\mathrm{coim}\, \mathrm{L}$ any chain $\mathrm{M}'_{u_0}-$ associated vectors of any vector $\varphi \in \ker \mathrm{L} \setminus \{0\}$ contains exactly p elements.

Denote by $\tilde{\mathrm{L}}$ the restriction of the operator L to $\mathrm{coim}\, \mathrm{L}$. By virtue of the Banach closed graph theorem the operator $\tilde{\mathrm{L}} : \mathrm{coim}\, \mathrm{L} \to \mathrm{im}\, \mathrm{L}$ is a toplinear isomorphism. Let $U_0^0 = \ker \mathrm{L}$ and construct sets $U_q^0 = \tilde{A}^q[U_0^0],\ q = \overline{1,p}$, where $\tilde{A} = \tilde{\mathrm{L}}^{-1}\mathrm{M}'_{u_0}$. Sets $U_q^0 \subset \mathrm{coim}\, \mathrm{L}$ are linear spaces, the image $F_p^0 = \mathrm{M}'_{u_o}[U_p^0]$ is a linear space, and $F_p^0 \cap \mathrm{im}\, \mathrm{L} = \{0\}$ (under condition **(C1)**).

Let us introduce one more condition

(C2). $F_p^0 \oplus \mathrm{im}\, \mathrm{L} = F$.

Equation (4) can be rewritten in the form

$$\mathrm{L}\dot{u} = \mathrm{M}'_{u_0} u + \mathrm{F}(u), \tag{5}$$

where $\mathrm{F} = \mathrm{M} - \mathrm{M}'_{u_0} \in C^{\infty}(U;F)$ by construction. Having influenced the equation (5) successively by the projectors $Q_q : F \to F_q^0$ $(F_q^0 = \mathrm{M}'_{u_0}[U_q^0],\ q = \overline{1,p})$ and $I - Q$ we obtain the equivalent system

$$\begin{cases} \mathrm{L}\dot{u}_1^0 = \mathrm{M}'_{u_0} u_0^0 + \mathrm{F}_0(u), \\ \dots \\ \mathrm{L}\dot{u}_p^0 = \mathrm{M}'_{u_0} u_{p-1}^0 + \mathrm{F}_{p-1}(u), \\ 0 = \mathrm{M}'_{u_0} u_p^0 + \mathrm{F}_p(u), \\ \mathrm{L}\dot{u}^1 = (I - Q)\mathrm{M}(u), \end{cases} \tag{6}$$

where $u_q^0 \in U_q^0,\ \mathrm{F}_q = Q_q \mathrm{F}(u) + Q_q \mathrm{M}'_{u_0} u^1,\ q = \overline{1,p},\ u^1 \in U^1$.

Lemma 1. *Let the operators* $\mathrm{L} \in L(U;F),\ \mathrm{M} \in C^{\infty}(U;F)$, *and* L *be a bi-splitting operator, and conditions (C1) and (C2) be satisfied. Then the equation (4) is equivalent to the system (6).*

Remark 1. Under the conditions of Lemma 1 the operator M'_{u_0} (L, p) is bounded at the point u_0 [5, 6].

Let us find solutions to the problem (3), (4). To obtain quasi-stationary trajectories from the set of possible solutions to the problem (3), (4), we introduce one more condition.

Let us consider the set $\tilde{U} = \{u \in U : u_q^0 = \text{const}, q = \overline{1, p}\}$. $\tilde{U}$ is a complete affine manifold, modeled by the subspace $U_0^0 \oplus U^1$. Let the point $u_0 \in \tilde{U}$, by $O_{u_o} \subset \tilde{U}$ we denote the neighborhood of the point u_0.

(C3). $F_q(u) \equiv 0 \quad \forall u \in O_{u_0}, \ q = \overline{1, p}.$

Theorem 1. *Let*

(i) *the conditions of Lemma 1 are satisfied;*

(ii) *point* $u_0 \in B$, *where* $B = \{u \in \tilde{U} : Q_0 \mathrm{M}(u) = 0\}$;

(iii) *condition (C3) is satisfied*

Then there is a unique solution of the problem (3), (4), which is a quasi-stationary trajectory, and $u(t) \in B \ \forall t \in (-t_0, t_0)$.

As in [7], we pass from the system (1) to the system

$$\begin{cases} (1 - æ\nabla^2) v_t = \nu \nabla^2 v - (v \cdot \nabla) v + \sum_{m=1}^{r} \sum_{s=0}^{n_m - 1} A_{m,s} \nabla^2 w_{m,s} - \overline{p}, \\ 0 = \nabla(\nabla \cdot v), \\ \dfrac{\partial w_{m,0}}{\partial t} = v + \alpha_m w_{m,0}, \qquad m = \overline{1, r}, \\ \dfrac{\partial w_{m,s}}{\partial t} = s w_{m,s-1} + \alpha_m w_{m,s}, \ s = \overline{1, n_m - 1}, \quad \alpha_m \in \mathbb{R}_-, \quad A_{m,s} \in \mathbb{R}_+ \end{cases} \tag{7}$$

We will be interested in the local unique solvability of the problem (7), (2).

Let us reduce problem (7), (2) to problem (3), (4). For this we set

$$U = \oplus_{l=0}^{K} U_l, \qquad F = \oplus_{l=0}^{K} F_l, \qquad K = n_1 + n_2 + \dots + n_r, \tag{8}$$

Where $U_0 = H_\sigma^2 \times H_\pi^2 \times H_p$, $F_0 = H_\sigma \times H_\pi \times H_p$; $U_i = H^2 \cap \overset{\circ}{H}{}^1 = H_\sigma^2 \times H_\pi^2$, $F_i = L^2 = H_\sigma \times H_\pi$, $i = \overline{1, K}$. Here H_σ^2 is the subspace of the solenoidal vectors of the space $H^2 \cap \overset{\circ}{H}{}^1$, $H^2 = (W_2^2(\Omega))^n$, $\overset{\circ}{H}{}^1 = (\overset{\circ}{W}{}_2^1(\Omega))^n$; H_π^2 is orthogonal (in the sense $L^2(\Omega) = (L^2(\Omega))^n$) complement to H_σ^2; H_σ and H_π – are the closures of the subspaces H_σ^2 and H_π^2 in the norm L^2 respectively, $H_p = H_\pi$ $\Sigma : L^2(\Omega) \to H_\sigma$ is the orthoprojector along H_π. Then $\Sigma \in L(H^2 \cap \overset{\circ}{H}{}^1)$, and $im\Sigma = H_\sigma^2, ker\Sigma = H_\pi^2$. The element of space U is a vector $\vec{u}(x,t)$, it has the form

$$\vec{u}(x,t) = (u_\sigma, u_\pi, u_p, w_{1,0}, \dots, w_{r,0}, w_{1,1}, \dots, w_{1l_1}, \dots, w_{r,1}, \dots, w_{rl_r}),$$

where $u_\sigma = \Sigma v$, $u_\pi = (I - \Sigma) v$, $u_p = \overline{p}$, $l_s = n_s - 1$, $s = \overline{1, r}$ and

$$\vec{u}(0) = (u_{\sigma_0}, u_{\pi_0}, u_{p_0}, w_{1,0}^0, \dots, w_{r,0}^0, w_{1,1}^0, \dots, w_{1,l_1}^0, \dots, w_{r,1}^0, \dots, w_{rl_r}^0),$$

where

$$u_{\sigma_0} = \Sigma v_0, \quad u_{\pi_0} = (I - \Sigma) v_0, \quad u_{p_0} = \overline{p}_0, \quad w_{i0}^0 = w_{i0}(x,0), \quad i = \overline{1, r}; \quad w_{ij}^0 = w_{ij}(x,0),$$

$$i = \overline{1, r}, \quad j = \overline{1, l_r}; \quad \vec{u}(x,t) = 0 \quad \forall (x,t) \in \partial\Omega \times \mathbb{R}.$$

Operators $L, M : U \to F$ are defined by formulas

$$L = \begin{pmatrix} \Sigma A_{æ}\Sigma & 0 & 0 & 0 & \cdots & 0 \\ 0 & \Pi A_{æ}\Pi & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I \end{pmatrix}, \tag{9}$$

where $\Pi = I - \Sigma$, $A_{æ} = 1 - æ\nabla^2$. Note that L is the order matrix $K+3$.

$$M(\vec{u}) = M_1\vec{u} + M_2(\vec{u}), \tag{10}$$

where M_1 is a matrix of the form

$$\begin{pmatrix} \nu\tilde{\Delta} & \nu\tilde{\Delta} & 0 & A_{10}\tilde{\Delta} & \cdots & A_{r0}\tilde{\Delta} & A_{11}\tilde{\Delta} & \cdots & A_{1l_1}\tilde{\Delta} & \cdots & A_{rl_r}\tilde{\Delta} \\ \nu\hat{\Delta} & \nu\hat{\Delta} & -I & A_{10}\hat{\Delta} & \cdots & A_{r0}\hat{\Delta} & A_{11}\hat{\Delta} & \cdots & A_{1l_1}\hat{\Delta} & \cdots & A_{rl_r}\hat{\Delta} \\ \Sigma C & \Pi C & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\ I & I & 0 & \alpha_1 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ I & I & 0 & 0 & \cdots & \alpha_r & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & 0 & I & \cdots & 0 & \alpha_1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \alpha_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & \alpha_r \end{pmatrix}.$$

Here $\tilde{\Delta} = \Sigma\Delta$, $\hat{\Delta} = \Pi\Delta$; $C(u_\sigma + u_\pi) = \nabla(\nabla \cdot (u_\sigma + u_\pi))$.

The operator M_2 has the form $M_2 = (\Sigma B(u_\sigma + u_\pi) + f_\sigma, \Pi B(u_\sigma + u_\pi) + f_\pi, 0, \ldots, 0)^T$, where $B(u_\sigma + u_\pi) = -((u_\sigma + u_\pi) \cdot \nabla)(u_\sigma + u_\pi)$.

Lemma 2. *Let spaces* U, F *be defined by formulas* (8), *and let* $n = 2, 3, 4$, *and operators* $L, M : U \to F$ *be defined by formulas* (9), (10). *Then:* (i) *operator* $L \in L(U; F)$, *and if* $æ^{-1} \notin \sigma(-\nabla^2)$, *then* $ker L = \{0\} \times \{0\} \times H_p \times \underbrace{\{0\} \ldots \{0\}}_{K}$, $im L = H_\sigma \times H_\pi \times \{0\} \times F_1 \times \ldots \times F_K$; ($ii$) *operator* $M \in C^\infty(U; F)$.

The statement (i) of Lemma 1 is obvious, and the statement (ii) is checked directly.

$$M'_u = M_1 + M_3, \tag{11}$$

where operator M_1 is defined above, and $M_3 = \begin{pmatrix} \Sigma B_\sigma & \Sigma B_\pi & 0 \\ \Pi B_\sigma & \Pi B_\pi & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Here $B_\sigma (B_\pi)$ is the Fréchet partial derivative of the operator B at the point $u_\sigma + u_\pi$ по $u_\sigma (u_\pi)$. Obviously, $\forall n \geq 3$ $\forall u \in U$ $M_u^{(n)} \equiv 0$ [7].

We have reduced the problem (7), (2) to the problem (3), (4).

Next, we check the feasibility of conditions (C1)–(C3). Denote by $A_{æ\sigma}$ the restriction of the operator $\Sigma A_{æ}\Sigma$ to H_σ^2.

Lemma 3. Let the conditions of Lemma 2 be satisfied, and $ker A_{æ\sigma} = \{0\}$. Then each vector $\varphi \in ker L \setminus \{0\}$ has exactly one M'_u is associated vector, regardless of the point $u \in U$.

Proof. Let the vector $\varphi = (0, 0, \varphi_p, 0, \ldots, 0) \in ker L$, $\varphi_p \neq 0$. Find the vector $\psi \in U$ such that $L\psi = M'_u\varphi$. From (9) and (10) we have

$$A_{æ\sigma}\psi_\sigma = 0, \qquad \Pi A_{æ}\psi_\pi = -\varphi_p. \tag{12}$$

We get $\psi_\sigma = 0$, $\psi_\pi = -\Pi A_{æ}^{-1}\varphi_p$, the component ψ_p of vector ψ is arbitrary, and the remaining K components of the vector ψ are equal to zero. $M'_u\psi \notin imL$, since $C\psi_\pi \neq 0$, if $\psi_\pi \neq 0$ [6]. •

The condition (C1) is satisfied for $p = 1$. Now let's check condition (C2). Denote by $A_{æ\pi}$ the restriction of the operator $\Pi A_{æ}^{-1}\Pi$ to H_π.

Lemma 4. *Under the conditions of Lemma 3, the operator* $A_{æ\pi} : H_\pi \to H_\pi^2$ *is a toplinear isomorphism.*

By virtue of Lemma 2, the operator L from (9) is bisplitting. Let $U_0^0 = kerL$, $coimL = H_\sigma^2 \times H_\pi^2 \times \{0\} \times U_1 \times \ldots \times U_K$.

Let's describe lineals

$$F_0^0 = M'_{u_0}[U_0^0] = \{0\} \times H_p \times \{0\} \times \underbrace{\{0\} \times \ldots \times \{0\}}_{K} = \{0\} \times H_\pi \times \{0\} \times \underbrace{\{0\} \times \ldots \times \{0\}}_{K} \subset imL,$$

$$U_1^0 = \tilde{L}^{-1}[F_0^0] = \Sigma A_{æ}^{-1}[H_p] \times A_{æ\pi}[H_p] \times \{0\} \times \underbrace{\{0\} \times \ldots \times \{0\}}_{K} =$$

$$\Sigma A_{æ}^{-1} A_{æ\pi}^{-1}[H_\pi^2] \times H_\pi^2 \times \{0\} \times \underbrace{\{0\} \times \ldots \times \{0\}}_{K} \subset coimL$$

by Lemma 4; $F_1^0 = M'_{u_0}[U_1^0] = \Sigma\tilde{B}_0 A_{æ}^{-1}[H_p] \times \Pi\tilde{B}_0 A_{æ}^{-1}[H_p] \times CA_{æ}^{-1}[H_p] \times \underbrace{\{0\} \times \ldots \times \{0\}}_{K}$.

Let $\tilde{C}$ be the restriction of the operator C to H_π^2. Since there is an operator $\tilde{C}^{-1}$ [6], then by Lemma 4

$$F_1^0 = \Sigma\tilde{B}_0 A_{æ}^{-1} A_{æ\pi}^{-1}\tilde{C}^{-1}[H_p] \times \Pi\tilde{B}_0 A_{æ}^{-1} A_{æ\pi}^{-1}\tilde{C}^{-1}[H_p] \times H_p \times \underbrace{\{0\} \times \ldots \times \{0\}}_{K} \not\subset imL.$$

Here and above $\tilde{B}_0$ is the Fréchet derivative of the operator $\tilde{B}$ at the point $u_{\sigma 0} + u_{\pi 0}$, and the operator $\tilde{L}^{-1}$ is defined from (9).

The operators P_0 and P_1 have the form of matrices

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \Pi & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & P_1^{12} & 0 & 0 & \cdots & 0 \\ 0 & \Pi & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{13}$$

where $P_1^{12} = \Sigma A_{æ}^{-1} A_{æ\pi}^{-1}\Pi$; and the operators are Q_0 and Q_1 respectively

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ Q_0^{21} & \Pi & Q_0^{23} & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & Q_1^{13} & 0 & \cdots & 0 \\ 0 & 0 & Q_1^{23} & 0 & \cdots & 0 \\ 0 & 0 & \Pi & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{14}$$

where $Q_1^{13} = \Sigma\tilde{B}_0 A_{æ}^{-1} A_{æ\pi}^{-1}\tilde{C}^{-1}\Pi$, $Q_1^{23} = \Pi\tilde{B}_0 A_{æ}^{-1} A_{æ\pi}^{-1}\tilde{C}^{-1}\Pi$, $Q_0^{21} = -\Pi A_{æ} A_{æ\sigma}^{-1}\Sigma$, $Q_0^{23} = -Q_0^{21}Q_1^{13} - Q_1^{23}$.

The operators $P_k \in L(U)$ and $Q_k \in L(F)$, $k = 0,1$, defined in (13), (14) are projectors, and $imP_k = U_k^0$, $im\,Q_k = F_k^0$, $k = 0,1$ and $P_0P_1 = P_1P_0 = 0$, $Q_0Q_1 = Q_1Q_0 = 0$. $ker\,Q_1 = imL$ and $F_1^0 \oplus im\,L = F$, condition (C2) is satisfied.

To check condition (C3) we construct the set

$$\tilde{U} = \{ u \in U : P_1 u = \text{const} \} = \{ u \in U : u_\pi = \text{const} \}.$$

The condition (C3) consists of the only equality $Q_1 M(u) = (Q_1^{13} C(u_\sigma + u_\pi), Q_1^{23} C(u_\sigma + u_\pi), C(u_\sigma + u_\pi), 0, \ldots, 0)^T = 0,$ which is executed if $u_\pi = 0$. If $\tilde{U} = \{u \in U : u_\pi = 0\}$, then the condition (C3) is implemented.

Let us construct the set *B*. According to Theorem 1, $B = \{u \in \tilde{U} : Q_0 M(u) = 0\}$. For $u_\pi = 0$ $Q_0 M(\vec{u}) = 0 \Leftrightarrow (Q_0^{21}\Sigma + \Pi)\tilde{B}(u_\sigma) - u_p = 0$, and

$$Q_0^{21}\Sigma + \Pi = A_{\ae\pi}^{-1}\Pi A_{\ae}^{-1}\Sigma + A_{\ae\pi}^{-1}\Pi A_{\ae}^{-1}\Pi = A_{\ae\pi}^{-1}\Pi A_{\ae}^{-1}, \tag{15}$$

then

$$B = \{u \in \tilde{U} : A_{\ae\pi}^{-1}\Pi A_{\ae}^{-1}(\tilde{B}(u_\sigma)) = u_p,\ u_\pi = 0, u_\sigma \in H_\sigma^2, u_i \in H_\sigma^2 \times H_\pi^2,\ i = \overline{1,K}\}. \tag{16}$$

Notice, that

$$\Pi A_{\ae}^{-1} A_{\ae\sigma}\Sigma + \Pi A_{\ae}^{-1}\Pi A_{\ae}\Sigma = \Pi A_{\ae}^{-1}(\Sigma A_{\ae} + \Pi A_{\ae})\Sigma = 0.$$

$$\Pi A_{\ae}^{-1} A_{\ae\sigma}\Sigma = -A_{\ae\pi}\Pi A_{\ae}\Sigma,\ A_{\ae\pi}^{-1}\Pi A_{\ae}^{-1} A_{\ae\sigma}\Sigma = -\Pi A_{\ae}\Sigma,\ A_{\ae\pi}^{-1}\Pi A_{\ae}^{-1}\Sigma = -\Pi A_{\ae} A_{\ae\sigma}^{-1}\Sigma = Q_0^{21}\Sigma.$$

Theorem 2. *Let the conditions of Lemma 3 be satisfied Let* $u_0 \in B$ (16)**.** *Then, for some* $t_0 = t_0(u_0)$ *, there exists the unique solution of the problem* (7), (2), *which is a quasistationary trajectory,* $u = (u_\sigma, 0, \bar{p}, w_{10}, \ldots, w_{r0}, w_{11}, \ldots, w_{1l_1}, \ldots, w_{r1}, \ldots, w_{rl_r})$ *of class* $C^\infty((-t_0, t_0); U)$, *and such that* $u \in B$ *for all* $t \in (-t_0, t_0)$.

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Received June 23, 2022

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Bulletin of the South Ural State University
Series "Mathematics. Mechanics. Physics"
2022, vol. 14, no. 3, pp. 45–51

УДК 517.958 **DOI: 10.14529/mmph220305**

АНАЛИЗ ОДНОГО КЛАССА ГИДРОДИНАМИЧЕСКИХ СИСТЕМ

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Аннотация. Рассмотрена разрешимость задачи Коши–Дирихле для обобщенной однородной модели динамики вязкоупругой несжимаемой жидкости Кельвина–Фойгта высокого порядка. При исследовании использована теория полулинейных уравнений соболевского типа. Указанная задача для системы дифференциальных уравнений в частных производных сводится к задаче Коши для указанного типа уравнения. Доказана теорема о существовании единственного решения указанной задачи, которое есть квазистационарная траектория, описано ее фазовое пространство.

Ключевые слова: уравнение соболевского типа; фазовое пространство; вязкоупругая несжимаемая жидкость.

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Поступила в редакцию 23 июня 2022 г.

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