

## SOUND WAVE PROPAGATION ALONG THE WEDGE EDGE

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There is given a theoretical research of Rayleigh wave's propagation in an elastic wedge. The dependencies of propagation coefficient got are in good correlation with well-known experimental data. There is given a theoretical research of Rayleigh wave's transformation of surface waves in an elastic wedge. The dependencies of transformation coefficient got are in good correlation with well-known experimental data.

**Formulation.** Let's assume that an elastic media characterized by Lamé constants  $\lambda$ ,  $\mu$  and density  $\rho_0$  fills infinite wedge shaped domain  $\Omega(-\theta_0, \theta_0) = (r \geq 0, -\theta_0 \leq \theta \leq \theta_0, z > 0)$ , where  $z$  – is directed along the wedge edge. Differential equation of particles' movement in absence of volume forces in the form of a vector could be represented in the form:

$$(\lambda + \mu)\text{grad div}U + \mu\Delta U = \rho_0 U'' . \quad (1)$$

Expressing vector potential in terms of 2 scalar functions  $\psi_1$  and  $\psi_2$ , let's express a displacement vector in the form:

$$U = \text{grad } \varphi + \text{rot } \psi_1 k + \text{rot rot } \psi_2 k \quad (2)$$

where  $\varphi$  – scalar potential;  $k$  – unit vector along the  $z$ -axis.

In this case, solution of a vector wave equation is separated for each function  $\psi_1$  and  $\psi_2$ :

$$\Delta \varphi + K_j \varphi = 0, \quad \Delta \psi_j + K_j \psi_j = 0, \quad j = 1, 2, \quad (3)$$

where

$$\Delta = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}, \quad K_h = \frac{\omega}{C_h},$$

here  $C_l, (C_t)$  – propagation speed of longitudinal and transverse waves,  $\omega$  – angular frequency.

No density condition  $\sigma_{\rho\rho} = \sigma_{\theta z} = \sigma_{\theta\theta} = 0$  is observed on free wedge boundaries  $\theta = \pm\theta_0$ . Let's also assume, that particles' movement starts from a state at rest.

It should be noted that to formulate a problem to a motion equation correctly, there should be boundary conditions, radiation conditions as well as a condition, regulating the behavior of the unknown functions in the vicinity of singular points, close to the edge.

This condition is equivalent to the requirement to observe the law of conservation of energy, and can be given in the following form:

$$U = O(\rho^p), \quad \rho \rightarrow 0$$

**2. Solution of the problem.** Let's study the solutions, corresponding to the harmonic waves running towards axis  $z$  in the form:

$$\varphi = (A_0 \text{ch } \nu_1 \theta + C_0 \text{sh } \nu_1 \theta) H_{\nu_1}^{(1)}(\rho \alpha) \exp i(pz - \omega t), \quad (4)$$

$$\psi_1 = (A_1 \text{ch } \nu_2 \theta + C_1 \text{sh } \nu_2 \theta) H_{\nu_2}^{(1)}(\rho \beta) \exp i(pz - \omega t),$$

where

$$\alpha = \sqrt{p^2 - K_l^2}, \quad \beta = \sqrt{p^2 - K_t^2}, \quad p > 0,$$

here  $p$  – wave number,  $\nu_1$  and  $\nu_2$  – angular wave numbers,  $J_{\nu_1}$  and  $J_{\nu_2}$  – Bessel functions.

Further, factor  $i(z - \omega t)$  is dropped.

Using (2) and Hook's Law for an isotrope body, it better to give tensor tensions' components in terms of wave potentials:

$$\frac{\sigma_{\theta\rho}}{\mu} = \frac{2}{\rho} \frac{\partial^2 \varphi}{\partial \rho \partial \theta} - \frac{2}{\rho^2} \frac{\partial \varphi}{\partial \theta} + \frac{1}{\rho} \frac{\partial \psi_1}{\partial \rho} - \frac{\partial^2 \psi_1}{\partial \rho^2} + \frac{1}{\rho^2} \frac{\partial^2 \psi_1}{\partial \theta^2} + \frac{2}{\rho} \frac{\partial^3 \psi_2}{\partial \rho \partial \theta \partial z} - \frac{2}{\rho^2} \frac{\partial^2 \psi_2}{\partial \theta \partial z},$$

$$\frac{\sigma_{\alpha z}}{\mu} = \frac{2}{\rho} \frac{\partial^2 \varphi}{\partial \theta \partial z} - \frac{\partial^2 \psi_1}{\partial \rho \partial z} + \frac{1}{\rho} \frac{\partial^3 \psi_2}{\partial z^2 \partial \theta} - \frac{1}{\rho^2} \frac{\partial^2 \psi_2}{\partial \rho \partial \theta} - \frac{1}{\rho^2} \frac{\partial^3 \psi_2}{\partial \theta^3} - \frac{1}{\rho} \frac{\partial^3 \psi_2}{\partial \theta \partial \rho^2}, \quad (5)$$

$$\frac{\sigma_{\theta\theta}}{2\mu} = \frac{\partial^2 \varphi}{\partial z^2} - \frac{K_t^2}{2} \varphi - \frac{\partial^2 \varphi}{\partial \rho^2} - \frac{\partial}{\partial \rho} \left( \frac{1}{\rho} \frac{\partial \psi_1}{\partial \theta} \right) + \frac{1}{\rho} \frac{\partial^2 \psi_2}{\partial \rho \partial z} + \frac{1}{\rho^2} \frac{\partial^3 \psi_2}{\partial \theta^2 \partial z}$$

Substituting expressions (4) with six constants  $A_0, A_1, A_2, C_0, C_1$  and  $C_2$  into (5) we obtain an algebraic system of six uniform equations to bind  $A_0, A_1, A_2, C_0, C_1$  and  $C_2$ .

This system has solutions corresponding to symmetric and antisymmetric with respect to plane  $\theta = 0$  waves.

A condition of existence of nontrivial solutions of this equation system is that determinant, which is represented in the form of a product  $\Delta_s \cdot \Delta_{as}$ , where  $\Delta_{as}$  corresponds to antisymmetric, and  $\Delta_s$  – to symmetric waves, is equal to zero. This equation has a solution at two independent conditions  $\Delta_s = 0$  and  $\Delta_{as} = 0$ , which determine two different kinds of surface waves. Let's consider the first kind of waves.

**3. Antisymmetric case.** Let's assume, that there are no disturbance oscillations in the vicinity of wave' propagation along  $\rho$  axis. Then, neglecting the derivatives of  $\rho$  coordinate, the equation for this case will have the following form:

$$\frac{(p^2 + \beta^2)^2}{4\alpha\beta p^2} - \left[ \frac{th v_2 \theta}{th v_1 \theta} \right]^b = 0, \quad (6)$$

$$v_1 = \sqrt{p^2 - K_t^2} \cdot \rho, \quad v_2 = \sqrt{p^2 - K_t^2} \cdot \rho$$

It's not difficult to see, that at big values of a wedge angle or a distance to its edge equation (6) transfers into the known Rayleigh equation.

For a complete solution in the boundary angular vicinity, it is necessary to include the condition on the edge:

$$U_\rho = \frac{\partial \varphi}{\partial \rho} + \frac{\partial^2 \psi_2}{\partial \rho \partial z} = const = A, \quad \rho \rightarrow \alpha, \quad (7)$$

where  $A$  – is constant.

Using the result of [5], it's possible to determine the value of displacements' amplitudes.

We'll use Fourier transformation on  $\rho$  coordinate for equation (3) and (5).

Then, the solution of the wave equation, simultaneously satisfying the solution on the edge and boundary conditions, can be given in the form:

$$\tilde{\varphi}(k_r) = \frac{AJ_{v_1}(K_t \rho)}{BJ'_{v_1}(K_t \rho) + iCJ'_{v_2}(K_t \rho)p}, \quad \tilde{\psi}(k_r) = \frac{AJ_{v_2}(K_t \rho)}{BJ'_{v_1}(K_t \rho) + iCJ'_{v_2}(K_t \rho)p},$$

where  $J_\nu(x)$  – Bessel function of the first kind,  $A, B, C$  – constants,  $B$  and  $C$  are interconnected by boundary conditions on the wedge surface, and primes denote derivatives of  $\rho$  coordinate.

Transferring inverse Fourier transformation into a complex plane, we obtain solutions, corresponding to residues in  $p_n$  poles of an integrand expression.

$$\varphi = \sum_{n=0}^{\infty} A_n e^{ip_n z}, \quad \psi = \sum_{n=0}^{\infty} B_n e^{ip_n z}, \quad (8)$$

where

$$A_n = \frac{J_{v_1}(K_t \rho)}{\left(\frac{\partial^2 J_{v_2}}{\partial \rho \partial p}\right)_{p=p_n}}, \quad B_n = \frac{J_{v_2}(K_t \rho)}{\left(\frac{\partial^2 J_{v_2}}{\partial \rho \partial p}\right)_{p=p_n}}.$$

This solution, called waveguide solution, forms a discrete field spectrum. Values of  $K_t \rho$  roots from the denominator in equation (6) could be called «width» of the waveguide. Values of phase velocity also depend on this parameter and are found from equation (8). Fig. 1 shows distributions of amplitudes of normal waves of the second and sixth order, correspondingly, with wedge angle  $\theta = 15^\circ$ .

Fig. 2 gives angular dependencies of antisymmetric wave's velocity of the second and sixth order.

Discussion of result as the analysis shows, disturbances with a very low velocity can propagate along the wedge edge. With an increase of a wedge angle, the velocity of these disturbances increases, and it's

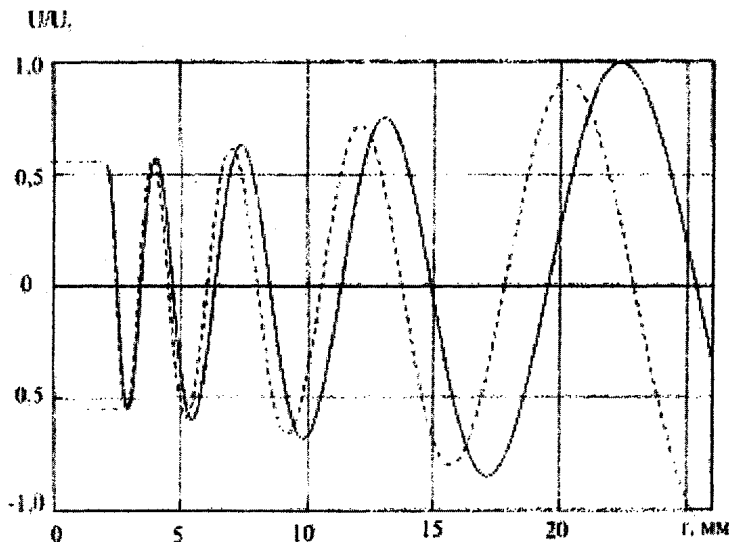


Fig. 1. Amplitudes of normal waves of the second and sixth order, correspondingly, with wedge angle  $\theta = 15^\circ$

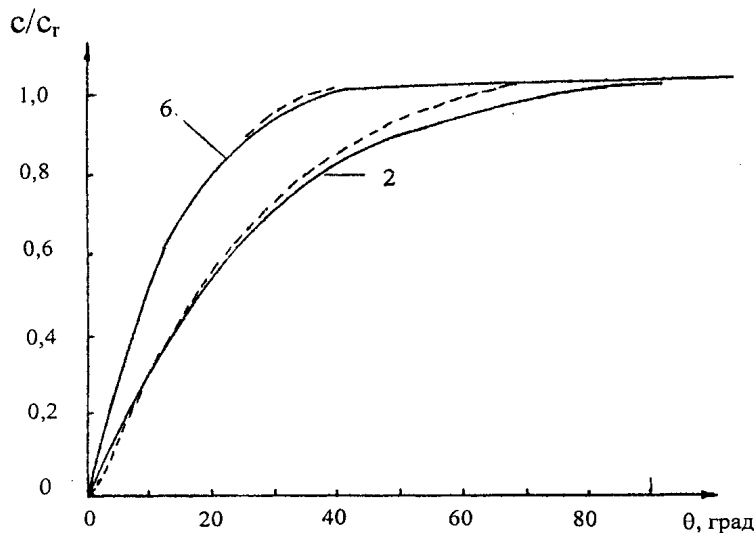


Fig. 2. Angular dependencies of antisymmetric wave's velocity

equal to velocity of Rayleigh wave at angles  $2\theta \geq 90^\circ$ . An increase of velocity of a wave with a wedge angle speaks about an increase of media's toughness, corresponding to a big local wedge thickness.

Propagation of an anti-symmetric wave along the edge takes place in a narrow wave channel. When disturbance diverge from the route of wave propagation into the media there happens a refraction of an antisymmetric wave because of a velocity decrease. As a result, there's observed a turn of rays towards the top of the wedge and, a return of waves to the wave channel. That's why this wave is acoustically stable in a narrow wave channel.

Propagation of a symmetric wave is of another character. Disturbance velocity of wedge media's particles, when approaching the edge increases monotonically up to the velocity of volume waves (see [6]). At a small divergence from the wave's propagation route the velocity of the wave decreases, and the path of disturbances is curving into the media. Thus, this wave is acoustically instable in a wave channel.

### References

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Поступила в редакцию 4 апреля 2003 г