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ON SOME CLASSES OF INVERSE PARABOLIC PROBLEMS OF RECOVERING THE THERMOPHYSICAL PARAMETERS

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Abstract. In the article we examine the question of regular solvability in Sobolev spaces of parabolic inverse coefficient problems. A solution is sought in the class of regular solutions that has all derivatives occurring in the equation summable to some power. The overdetermination conditions are the values of a solution at some collection of points lying inside the domain. The proof is based on a priori estimates and the fixed point theorem.

Keywords: parabolic equation, inverse problem, initial-boundary value problem, existence, uniqueness.

Introduction

We consider the well-posedness questions of inverse parabolic problems. Let G be a domain in $\mathbb{R}^n$ with boundary $\Gamma \in C^2$. The parabolic equation is of the form

$$c(t,x)u_t + A(t,x,D)u = f(t,x),\ (t,x) \in Q = (0,T)\times G, \tag{1}$$

where the functions f, c and the elliptic operator A are as follows:

$$-A(t,x,D) = A_0(t,x,D_x) + \sum_{i=1}^{r} q_i(t) A_i(t,x,D_x),\ f = f_0(t,x) + \sum_{i=r_1+1}^{s} f_i(t,x) q_i(t),$$

$$A_i = \sum_{k,l=1}^{n} a_{kl}^i(t,x)\partial_{x_k x_l} + \sum_{k=1}^{n} a_k^i(t,x)\partial_{x_k} + a_0^i, c = c_0(t,x) - \sum_{i=r+1}^{r_1} f_i(t,x) q_i(t).$$

The equation (1) is furnished with the initial and boundary conditions

$$u_{t=0} = u_0, Bu|_S = g(t,x), S = (0,T)\times\Gamma, \tag{2}$$

where $Bu = u$ or $Bu = \frac{\partial u}{\partial N} + \sigma u = \sum_{i,j=1}^{n} a_{ij} u_{x_i} \nu_j + \sigma u$ ($\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ is the outward unit normal to Γ, and the overdetermination conditions

$$u(t,b_j) = \psi_j(t), b_j \in G, j = 1,2,\ldots,s. \tag{3}$$

The unknowns in (1)–(3) are a solution u and the functions $q_i(t)$ ($i = 1,2,\ldots,s$). The problem (1)–(3) arise in describing the heat and mass transfer, diffusion, and filtration processes, ecology, and many other fields. The problem of determining thermophysical and mass transfer characteristics with the use of inverse modelling is studied in [1], where the results are used for describing the temperature regimes of the soils of the northern territories. We can refer to the monograph [2] devoted to inverse parabolic probems and to [3–6], where the main statements of inverse problems and some applications can be found. The number of theoretical results devoted to the problems (1)–(3) is sufficiently small. We should refer to the articles [7–10], where in the case of $n = 1$ the thermal conductivity depending on time is defined and existence and uniqueness theorems are established with the additional data being the values of a solution at some points lying in the domain or on its boundary. The thermal conductivity independent of one of the spacial variable and some other coefficients are identified in [11, 12] with the use of the Cauchy data on the lateral boundary of the cylinder and integral data. Existence of a solution is proven and stability estimates are exposed. The monograph [3] (see also the results in [13]) contains the existence theory for inverse problems of recovering the coefficients in the leading part of the equation independent of some part of variables with the overdetermination data given on sections of the spacial domain by planes. In view of the method, all coefficients also are independent of some spacial variables. More complete results for the problems (1)–(3) can be found in [14–17], where the well-posedness of the inverse problems in question is established for the case of the additional data are the values of a solution on some spacial manifolds or at some collection of points. However, in these articles $c(t,x)=1$ except for the article [15], where $c(t,x)=c(t)$ in the case of the pointwise ovedetermination. The existence and uniqueness theorems in the case of the unknown heat capacity and the integral overdetermination data are exposed in [18–20], where $c(t,x)=c(t)$ or $c(t,x)=\text{const}$. Note that inverse problems with pointwise data have been studied by A.I. Prilepko and his followers and a number of classical results is presented in [2]. Similar results under different conditions on the data and in some other spaces can be found in [21, 22]. Our results are close to those in [23]. In contrast to this article, the heat capacity here is unknown. The main results of the article are exposed in Sect. 2.

Prelimiraries

The definition of the Sobolev spaces $W_p^s(G;E)$, $W_p^s(Q;E)$ (E is a Banach space) can be found in [24]. If $E=R$ or $E=R^n$ then we omit the notation E and write $W_p^s(Q)$. The definitions of the Hölder spaces $C^{\alpha,\beta}(\overline{Q}),C^{\alpha,\beta}(\overline{S})$ can be found in [25]. By the norm of a vector, we mean the sum of the norms of its coordinates. Given the interval $J=(0,T)$, put $W_p^{s,r}(Q)=W_p^s(J;L_p(G))\cap L_p(J;W_p^r(G))$, $W_p^{s,r}(S)=W_p^s(J;L_p(\Gamma))\cap L_p(J;W_p^r(\Gamma))$. All function spaces as well as the coefficients of the equation (1) are assumed to be real. In what follows, we suppose that $p>n+2$. The definition of the boundary of class C^s, $s\ge 1$, can be found in Ch. 1 in [25]. Denote by $B_\delta(b)$ the ball of radius δ centered at b. Fix a parameter $\delta>0$ such that $\overline{B_\delta(b_i)}\cap\overline{B_\delta(b_j)}=\varnothing$ for $i\ne j$ and $\overline{B_\delta(b_i)}\cap\Gamma=\varnothing$, $i,j=1,2,\ldots,s$. Denote $Q^\tau=(0,\tau)\times G$ and $G_\delta=\cup_i B_\delta(b_i)$. Construct nonnegative functions $\varphi_j\in C^\infty(\mathbb{R}^n)$ such that $\varphi_j=1$ in $B_{\delta/2}(b_j)$, $\varphi(x)\ge 0$ and $\varphi_j=0$ for $x\notin B_{3\delta/4}(b_j)$. Let $\varphi=\sum_{j=1}^{s}\varphi_j(x)$.

The consistency and smoothness conditions can be written as

$$u_0(x)\in W_p^{2-2/p}(G^\pm),B(0,x,D)u_0|_\Gamma=g(0,x),g\in W_p^{k_0,2k_0}(S), \tag{4}$$

where $k_0=s_1=1-1/2p$ for $Bu=u$ and $k_0=s_0=1/2-1/2p$ otherwise;

$$\varphi u_0(x)\in W_p^{3-\frac{2}{p}}(G),a_{ij}^k\in L_\infty(0,T;W_p^1(G_\delta)),a_l^k\in L_p(0,T;W_p^1(G_\delta)), \tag{5}$$

where $i,j=1,2,\ldots,n,l=0,1,\ldots,n,k=0,1,\ldots,r$;

$$f\in C(\overline{Q})\cap L_\infty(0,T;W_\infty^1(G_\delta)),\varphi f_l\in L_p(0,T;W_p^1(G)),f_l\in L_p(Q), \tag{6}$$

where $j=r+1,\ldots,r_1$ и $l=0,\ r_1+1,\ldots,s$. We use the inclusions of the form $f\in L_p(0,T;W_p^1(G_\delta))$ or similar, where the set G_δ consists of several connectedness components (in this case $B_\delta(b_j)$). By definition, this means that $f|_{B_\delta(b_j)}\in L_p(0,T;W_p^1(B_\delta(b_j)))$ for all j. This space is endowed with the norm equal the sum of the norms over the corresponding connectedness components We assume that

$$a_{ij}^k\in C(\overline{Q}),a_l^k\in L_p(Q),\sigma,a_{ij}^k|_S\in W_p^{s_0,2s_0}(S), \tag{7}$$

where the last inclusion is required only if $Bu\ne u$,

$$\psi_j\in C^1([0,T]),\psi_j(0)=u_0(b_j),a_l^k(t,b_j),f_m(t,b_j)\in C([0,T]), \tag{8}$$

where $j=1,\ldots,s,m=0,r+1,\ldots,s,l=0,1,\ldots,n,k=0,1,\ldots,r.$ In view of (5), (8), the traces $f_m(t,b_p),a_l^k(t,b_j)$ are defined and $f_m(t,b_p),a_l^k(t,b_j)\in L_p(0,T)$; moreover,

$f_m(t,x), a_l^k(t,x) \in C(\overline{G_\delta}; L_p(0,T))$ (after a possible change on a set of zero measure 0) (see [26], Sect. 2,3,4, the relations (3.1)–(3.9), the corollary 4.3).

Consider the matrix B_0 of dimension $s \times s$ with rows

$$A_1(0,b_j,D)u_0(b_j),\ldots,A_r(0,b_j,D)u_0(b_j), f_{r+1}(0,b_j)\psi_j'(0),\ldots, f_{r_1}\psi_j'(0), f_{r_1+1}(0,b_j),\ldots, f_s(0,b_j).$$

and assume that

$$\det B_0 \neq 0. \tag{9}$$

Consider the system

$$B_0\vec{q}_0 = \vec{g}_0, \tag{10}$$

$$\vec{g}_0 = (c_0(0,b_1)\psi_{1t}(0) - A_0(0,b_1,D)u_0 - f_0(0,b_1),\ldots,c_0(0,b_s)\psi_{st}(0) - A_0(0,b_s,D)u_0 - f_0(0,b_s))^T.$$

In view of (9) the system (10) has a unique solution $q_0 = (q_{01},\ldots,q_{0s})$. Denote $a_{pl} = a_{pl}^0 + \sum_{i=1}^{r} a_{pl}^i q_{0i}$ and below we suppose that

$$L_0(\xi) = \sum_{p,l=1}^{n} a_{pl}(t,x)\xi_p\xi_l \geq \delta_0 |\xi|^2 \ \forall \xi \in \mathbb{R}^n, \forall (t,x) \in Q.$$

$$C = c_0 - \sum_{i=r+1}^{r_1} q_{0i} f_i(t,x) \geq \delta_0 \ \forall (t,x) \in Q,$$

where δ_0 is a positive constant. The operator $-A^0 = A_0(t,x,D_x) + \sum_{i=1}^{r} q_{0i}A_i(t,x,D_x)$ is elliptic and we can consider the problem

$$C(t,x)u_t + A^0(t,x,D_x)u = f,\ u|_{t=0} = u_0(x), Bu|_S = g. \tag{11}$$

Theorem 1. *Assume that the conditions* (4), (7) *hold and* $f \in L_p(Q)$. *Then there exists a unique solution* $u \in W_p^{1,2}(Q)$ *to the problem* (11). *If* $g = 0$ *then it satisfies the estimate*

$$\|u\|_{W_p^{1,2}(Q^\tau)} \leq c\left[\|u_0\|_{W_p^{2-2/p}(G)} + \|f\|_{L_p(Q^\tau)}\right], \tag{12}$$

where the constant c *is independent of* $u_0, f, \tau \in (0,T]$. *If additionally the condition* (5) *holds and* $\varphi f \in L_p(0,T;W_p^1(G_\delta))$ *then* $\varphi u \in L_p(0,T;W_p^3(G))$, $\varphi u_t \in L_p(0,T;W_p^1(G))$ *and if* $g = 0$ *then*

$$\|u\|_{W_p^{1,2}(Q^\tau)} + \|\varphi u\|_{L_p(0,\tau;W_p^3(G))} + \|\varphi u_t\|_{L_p(0,\tau;W_p^1(G))} \leq c[\|u_0\|_{W_p^{2-2/p}(Q^\tau)} +$$
$$+\|\varphi u_0\|_{W_p^{3-2/p}(G)} + \|f\|_{L_p(Q^\tau)} + \|\varphi f\|_{L_p(0,\tau;W_p^1(G))}], \tag{13}$$

where c *is independent of* $u_0, f, \tau \in (0,T]$.

Proof. The first claim results from Theorem 2.1 in [24]. The estimate (12) results from the conventional arguments (see, for instance, Theorem 2 in [22], Theorem 1 in [21]). Additional smoothness of a solution is established as in Theorem 1 in [27] (see also the proof of theorem 4, subsect. 3, sect. 2, Ch. 4 in [23]). The claim is also contained in Theorem 1 in [28] which can be applied here.

Denote the left-hand side of (13) by $\|u\|_{H^\tau}$ and the quantity $\|f\|_{L_p(Q^\tau)} + \|\varphi f\|_{L_p(0,\tau;W_p^1(G))}$ by $\|f\|_{W^\tau}$. The corresponding Banach spaces are denoted by H^τ and W^τ, respectively. The space H^τ comprises the functions $u \in W_p^{1,2}(Q^\tau)$ such that $\varphi u \in L_p(0,T;W_p^3(G))$, $\varphi u_t \in L_p(0,T;W_p^1(G))$, u satisfies the homogeneous initial-boundary conditions in (11).

Main results

Theorem 2. *Let the conditions* (4)–(9) *hold. Then there exists a number* $\tau_0 \in (0,T]$ *such that on* $(0,\tau_0)$ *there exists a unique solution* $(u,q_1,q_2,\ldots,q_s)$ *to the problem* (1)–(3) *such that* $u \in W_p^{1,2}(Q^{\tau_0})$, $\varphi u \in L_p(0,\tau_0;W_p^3(G))$, $\varphi u_t \in L_p(0,\tau_0;W_p^1(G))$, $q_j \in C([0,\tau_0])$, $j=1,2,\ldots,s$.

Proof. Let $\vec{q}=(q_1,...,q_s)^T$. Find a solution Φ to the problem (11), where we take $f = f_0 + \sum_{i=r_1+1}^{s} f_i(t,x)q_{0i}$ and the functions g,u_0 are our data in (11). By Theorem 1, there exists a solution to the problem (11) such that $\Phi \in W_p^{1,2}(Q)$, $\varphi\Phi \in L_p(0,T;W_p^3(G))$, $\varphi\Phi_t \in L_p(0,T;W_p^1(G))$. Make the change $u=v+\Phi$. We obtain the problem

$$Lv = c(t,x)v_t + S(\vec{\mu})v = \left(A^0 - A\right)\Phi + \left(-c(t,x)+C(t,x)\right)\Phi_t + \sum_{i=r_1+1}^{s} f_i(t,x)\mu_i(t), \tag{14}$$

where $(\vec{\mu}) = -(A^0 + A(\vec{\mu}))$, $A(\vec{\mu}) = \sum_{i=1}^{r} \mu_i(t)A_i(t,x,D_x)$, $c = C - c^0(\vec{\mu},t,x)$, $c^0(\vec{\mu},t,x) = \sum_{i=r+1}^{r_1} f_i\mu_i$, $\mu_i(t) = q_i(t) - q_{0i}$, $\vec{\mu} = (\mu_1,\ldots,\mu_s)$;

$$v_{t=0} = 0, Bv_S = 0, v(t,b_j) = \psi_j(t) - \Phi(t,b_j) = \tilde{\psi}_j, j=1,\ldots,s. \tag{15}$$

In view of the properties of the function Φ, $D^\alpha\varphi\Phi \in W_p^{1,2}(Q)$ for all j and $|\alpha|\le 1$. The embedding theorems yield $D^\alpha\varphi\Phi_i(t,x) \in C^{1-(n+2)/2p,2-(n+2)/p}(\bar{Q})$ (see Sect. 6.3 and Theorem 1 (the Sect. Remarks p. 424) in [29]) and thereby $D^\alpha\Phi(t,b_j) \in C([0,T])$. In this case the function $a_{ij}^k(t,b_j)D_x^\alpha\Phi(t,b_j)$, $a_i^k(t,b_j)D_x^\alpha\Phi(t,b_j)$ belong to $C([0,T])$. Hence, $A^0\Phi(t,b_j) \in C([0,T])$ (after a possible change on a set of zero measure). Similarly, $C(t,b_j) \in C([0,T])$. Consider the right-hand side in the equation for Φ. We have that $f_k(t,b_j) \in C([0,T])$ (in view of (6), (8)). From the equation we infer $\Phi_t(t,b_j) \in C([0,T])$ for all j. Thus, we have reduced the problem (1)–(3) to a simpler problem (14)–(15). Let $B_{R_0} = \{\vec{\mu} \in C([0,\tau]) : \|\vec{\mu}\|_{C([0,\tau])} \le R_0\}$. Consider the expression $L(\vec{\xi}) = \sum_{ij=1}^{n} \tilde{a}_{ij}\xi_i\xi_j$, $\tilde{a}_{ij} = \sum_{k=1}^{r} a_{ij}^k\mu_k$ and find the quantity R_0 such that

$$\left|L(\vec{\xi})\right| \le \delta_0|\xi|^2/2 \ \forall\xi \in \mathbb{R}^n, |c-C| = \left|\sum_{i=r+1}^{r_1} \mu_i f_i\right| \le \delta_0/2, \forall(t,x) \in Q, \forall\vec{\mu} \in B_{R_0}.$$

In this case the operator $S(\vec{\mu})$ is elliptic and Theorem 1 holds with $S(\vec{\mu})$ rather than A^0. Given a vector $\vec{\mu} \in B_{R_0}$, find a solution v to the problem (1)–(2) on $(0,\tau)$ such that $\varphi v \in L_p(0,\tau;W_p^3(G))$, $\varphi v_t \in L_p(0,\tau;W_p^1(G))$. Study the properties of this map $\vec{\mu} \to v(\vec{\mu})$. Theorem 1 yields

$$v = L^{-1}f, \ f = \sum_{i=1}^{r} \mu_i A_i\Phi(t,x) + \sum_{i=r+1}^{r_1} \mu_i f_i\Phi_t + \sum_{i=r_1+1}^{s} \mu_i f_i(t,x). \tag{16}$$

We have the estimate

$$\|v\|_{H^\tau} = \left\|L^{-1}f\right\|_{H^\tau} \le c\|f\|_{W^\tau}, \tag{17}$$

The conditions on the coefficients imply the estimate

$$\|f\|_{W^\tau} \le c_2 \|\vec{\mu}\|_{C([0,\tau])}, \tag{18}$$

where the constant c_2 depends on the quantities $\|f_i\|_{W^\tau}$ $(i \ge r_1+1)$, $\|f_i\|_{L_\infty(0,\tau;W^1_\infty(G_\delta))}$ $(r+1 \le i \le r_1)$, $\|\Phi\|_{H^\tau}$ (we can replace τ with T in these norms and thus we can take c_2 independent of τ). Take $\vec{\mu}_i \in B_{R_0}$ $(i=1,2)$ and consider the corresponding solutions v_1, v_2 to the problem (14)–(15). Let $\vec{\mu}_i = (\mu_{1i}, \mu_{2i}, \ldots, \mu_{si}), i=1,2$. Subtracting the latter equation (1) from the former, we obtain that the difference $\omega = v_2 - v_1$, $v_i = v(\vec{\mu}_i)$ meets the equation

$$\frac{(c_1+c_2)}{2}\omega_t + S\left(\frac{\vec{\mu}_1+\vec{\mu}_2}{2}\right)\omega = \sum_{j=1}^{r}(\mu_{j2}(t)-\mu_{j1}(t))A_j(t,x,D)(v_1+v_2)/2 +$$

$$\sum_{j=r+1}^{r_1}(\mu_{j2}(t)-\mu_{j1}(t))f_j(t,x)(v_{1t}+v_{2t})/2 + \sum_{j=1}^{r}(\mu_{j2}(t)-\mu_{j1}(t))A_j(t,x,D)\Phi$$

$$+\sum_{j=r+1}^{r_1}(\mu_{j2}(t)-\mu_{j1}(t))f_j\Phi_t + \sum_{j=r_1+1}^{s} f_j(t,x)(\mu_{j2}(t)-\mu_{j1}(t)) = f. \tag{19}$$

We have that $(\vec{\mu}_1+\vec{\mu}_2)/2 \in B_{R_0}$ and, thus, the following estimate (see (17)) holds:

$$\|\omega\|_{H^\tau} \le c\|\tilde{f}\|_{W^\tau}, \tag{20}$$

The estimates (18), (20) ensure that

$$\|\omega\|_{H^\tau} \le c\|\tilde{f}\|_{W^\tau} \le c_2 c\|\vec{\mu}_2 - \vec{\mu}_1\|_{C([0,\tau])}, \tag{21}$$

where c_2 depends on the norms $\|(v_1+v_2)/2\|_{H^\tau}$, $\|f_i\|_{W^\tau}$ $(i \ge r_1+1)$, $\|f_i\|_{L_\infty(0,\tau;W^1_\infty(G_\delta))}$ $(r+1 \le i \le r_1)$, $\|\Phi\|_{H^\tau}$. Let $v = v(\vec{\mu})$ be a solution to the problem (14)–(15). Taking $x = b_j$ in (1) and taking into account that $v_t(t,b_j) = \tilde{\psi}_j'$, we obtain the system

$$c(t,b_j)\tilde{\psi}_j' + S(\vec{\mu})v(t,b_j) = \sum_{i=1}^{r}\mu_i A_i(t,b_j,D)\Phi(t,b_j) + \sum_{i=r+1}^{r_1}\mu_i f_i \Phi_t(t,b_j) + \sum_{i=r_1+1}^{s} f_i(t,b_j)\mu_j(t). \tag{22}$$

The right-hand side can be written as $B(t)\vec{\mu}$, where the rows of the matrix $B(t)$ are as follows:

$$A_1(t,b_j,D)\Phi(t,b_j),\ldots,A_r(t,b_j,D)\Phi(t,b_j), f_{r+1}\Phi_t(t,b_j),\ldots, f_{r_1}\Phi_t(t,b_j), f_{r_1+1}(t,b_j),\ldots, f_s(t,b_j).$$

The matrix $B(0)$ agrees with B_0 from (9) and thereby $\det B(0) \ne 0$. The functions $f_i(t,b_j), a^i_{kl}(t,b_j)$, $a^i_k(t,b_j)$ are continuous for all values of the indices. Moreover, $D^\alpha \varphi\Phi(t,x) \in C(\overline{Q})$ for $|\alpha| \le 2$. Thus, the entries of B are continuous in t and there exists $\tau_0 \le T$ and a constant $\delta_3 > 0$ such that

$$|\det B(t)| \ge \delta_3 > 0 \, \forall t \in [0,\tau_0]. \tag{23}$$

In this case the system (22) is written in the form

$$\vec{\mu}(t) = B^{-1}H(\vec{\mu})(t) = R(\vec{\mu}), H(\vec{\mu}) =$$

$$(c(t,b_1)\tilde{\psi}_1' + S(\vec{\mu})v(t,b_1), c(t,b_2)\tilde{\psi}_2' + S(\vec{\mu})v(t,b_2),\ldots, c(t,b_s)\tilde{\psi}_s' + S(\vec{\mu})v(t,b_s))^T \tag{24}$$

The right-hand side here contains an operator taking the vector $\vec{\mu}$ into the vector with the components $C(t,b_j)\tilde{\psi}_j' - c^0(\vec{\mu},t,b_j)\tilde{\psi}_j' + S(\vec{\mu})v(t,b_j)$ $(j=1,2,\ldots,s)$, where v is a solution to the problem (14), (15). The properties of the map $\vec{\mu} \to v(\vec{\mu})$ we have already studied. Demonstrate that there exists $\tau_1 \le \tau_0$ such that the operator $R(\vec{\mu}) = B^{-1}H(\vec{\mu})(t)$, $R: C([0,\tau_1]) \to C([0,\tau_1])$ takes the ball B_{R_0} into itself and is a contraction. Consider the quantity $\psi_j^{'}(0)$. By construction,

$$C(0,b_j)\tilde{\psi}_j^{'}(0) = C(0,b_j)\left(\psi_j^{'}(0) - \Phi^{'}(0,b_j)\right) = C(0,b_j)\psi_j^{'}(0) + A(0,b_j,D)u_0(b_j) -$$

$$f_0 - \sum_{i=r_1+1}^{s} f_i(0,b_j)q_{0i} = 0,\ j = 1,\dots,s,$$

since the numbers q_{0i} are defined from (10). Let

$$\vec{\psi} = (C(0,b_1)\tilde{\psi}_1^{'}, C(0,b_2)\tilde{\psi}_2^{'},\dots,C(0,b_s)\tilde{\psi}_s^{'})^T.$$

In this case $\vec{\psi} \in C([0,\tau])$ ($\tau \le \tau_0$) and $\vec{\psi}(0) = 0$. There exists a number $\tau_1 \le \tau_0$ such that $\left\| B^{-1}(t)\vec{\psi} \right\|_{C([0,\tau_1])} \le R_0/2$. Note that $R(0) = B^{-1}(t)\vec{\psi}(t)$. Next, we obtain estimates assuming that $\vec{\mu} \in B_{R_0}$ and $\tau \le \tau_1$. In this case $\vec{\psi} \in C([0,\tau])$ ($\tau \le \tau_0$) and $\vec{\psi}(0) = 0$. There exists a number $\tau_1 \le \tau_0$ such that $\left\| B^{-1}(t)\vec{\psi} \right\|_{C([0,\tau_1])} \le R_0/2$. Note that $R(0) = B^{-1}(t)\vec{\psi}(t)$. Next, we obtain estimates assuming that $\vec{\mu} \in B_{R_0}$ and $\tau \le \tau_1$. We have

$$\left\| R(\vec{\mu}_1) - R(\vec{\mu}_2) \right\|_{C([0,\tau])} \le \sum_{j=1}^{S}\sum_{i=r+1}^{r_1} \left\| (\mu_{1i} - \mu_{2i}) f_i(t,b_j)\tilde{\psi}_j' \right\|_{C([0,\tau])} +$$

$$\sum_{i=1}^{s} \left\| A_0 v_1(t,b_i) - A_0 v_2(t,b_i) \right\|_{C([0,\tau])} + \sum_{i=1}^{s}\sum_{k=1}^{r} \left\| \mu_{1k} A_k v_1(t,b_i) - \mu_{2k} A_k v_2(t,b_i) \right\|_{C([0,\tau])}. \tag{25}$$

Next, we employ the conditions on the data and the embedding $W_p^{\theta}(G) \subset C(\overline{G})$ for $\theta > n/p$ (see Theorems 4.6.1,4.6.2 in [30]). Take $\theta \in (n/p, 1-2/p)$. Consider the last summand. We have

$$\left\| \mu_{1k} A_k v_1(t,b_i) - \mu_{2k} A_k v_2(t,b_i) \right\|_{C([0,\tau])} \le \left\| (\mu_{1k} - \mu_{2k})(A_k v_1(t,b_i) + A_k v_2(t,b_i)) \right\|_{C([0,\tau])} / 2 +$$

$$+ \left\| \frac{(\mu_{1k} + \mu_{2k})}{2} (A_k(v_1(t,b_i) - v_2(t,b_i)) \right\|_{C([0,\tau])} \le$$

$$\left\| \mu_{1k} - \mu_{2k} \right\|_{C([0,\tau])} c_4 \sum_{|\alpha| \le 2} \left\| D^{\alpha}\left(v_1(t,b_j) + v_2(t,b_j)\right) \right\|_{C([0,\tau])} +$$

$$\left\| \mu_{1k} + \mu_{2k} \right\|_{C([0,\tau])} c_5 \sum_{|\alpha| \le 2} \left\| D^{\alpha} v_1(t,b_j) - D^{\alpha} v_2(t,b_j) \right\|_{C([0,\tau])} \le$$

$$\left\| \mu_{1k} + \mu_{2k} \right\|_{C([0,\tau])} c_7 \left\| \varphi\left(v_1(t,x) - v_2(t,x)\right) \right\|_{C\left([0,\tau]; W_p^{2+\theta}(G)\right)}, \tag{26}$$

where the constant c_i are independent of τ. Let $v \in H^{\tau}$. In this case $\varphi v \in L_p(0,\tau; W_p^3(G))$, $\varphi v_t \in L_p(0,\tau; W_p^1(G))$ and thereby $\varphi v \in C\left([0,\tau]; W_p^{3-2/p}(G)\right)$ (see Theorem III 4.10.2 in [29] and [30]). The inequality

$$\left\| \varphi v \right\|_{C\left([0,\tau]; W_p^{3-2/p}(G)\right)} \le c_8 \left(\left\| \varphi v \right\|_{L_p\left(0,\tau; W_p^3(G)\right)} + \left\| \varphi v_t \right\|_{L_p\left(0,\tau; W_p^1(G)\right)} \right), \tag{27}$$

is valid, where the constant c_8 is independent of $\tau \in (0,T]$. Indeed, consider the function $w(t,x) = v(t,x)$ for $t \le \tau$, $w(t,x) = v(2\tau - t, x)$ for $t \in (\tau, 2\tau)$ and $w(t,x) = 0$ for $t \ge 2\tau$. We use the fact that $v(0,x)=0$. Next, we write out (27) on $[0,T]$ for the function w, the constant in this inequality is independent of τ. Next, we estimate the right-hand side of the inequality obtained from above with the use of the definition of w and obtain that the constant c_8 in (27) is independent of τ. Let $\theta_1(3-2/p)=2+\theta$. We have that $\theta_1\in(0,1)$. Denote $\omega=v_2-v_1$. Next, we use the interpolation properties of the Sobolev spaces [30] and (27). We infer

$$\|\varphi\omega\|_{C\left([0,\tau];W_p^{2+\theta}(G)\right)}\le c\|\varphi\omega\|^{\theta_1}_{C\left([0,\tau];W_p^{3-2/p}(G)\right)}\|\varphi\omega\|^{1-\theta_1}_{C\left([0,\tau];L_p(G)\right)}\le$$

$$c_1\tau^{\gamma}\|\varphi\omega_t\|^{1-\theta_1}_{L_p\left(0,\tau;L_p(G)\right)}\|\varphi\omega\|^{\theta_1}_{H^\tau}\le c_2\tau^{\gamma}\|\varphi\omega\|_{H^\tau},\ \gamma=(1-1/p)(1-\theta_1), \tag{28}$$

where we employ the obvious inequality

$$\|v\|_{C\left([0,\tau];L_p(G)\right)}\le\tau^{1-1/p}\|v_t\|_{L_p\left(0,\tau;L_p(G)\right)},\ v(0)=0.$$

The inequality (28) yields

$$\|\varphi\omega\|_{C\left([0,\tau];W_p^{2+\theta}(G)\right)}\le c_9\tau^{\gamma}\|\varphi\omega\|_{H^\tau}, \tag{29}$$

where the constant c_9 is independent of τ. Аналогично получим

$$\|\varphi(v_1+v_2)\|_{C\left([0,\tau];W_p^{2+\theta}(G)\right)}\le c_9\tau^{\gamma}\|\varphi(v_1+v_2)\|_{H^\tau}.$$

In view of the inequalities (17), (18) (written for the functions v_i), (21), (26), and (29), we conclude that

$$\|\mu_{1k}A_kv_1(t,b_i)-\mu_{2k}A_kv_2(t,b_i)\|_{C([0,\tau])}\le c_{13}\tau^{\gamma}\|\mu_{1k}-\mu_{2k}\|_{C([0,\tau])}, \tag{30}$$

where c_{12} is independent of $\tau\le\tau_1$ (it depends on R_0). Similar estimate holds for the second summand in (25). The first summand is estimated by

$$\sum_{j=1}^{s}\sum_{i=r+1}^{r_1}\left\|(\mu_{1i}-\mu_{2i})f_i(t,b_j)\tilde\psi_j'\right\|_{C([0,\tau])}\le c_{14}\left\|\vec\mu_1-\vec\mu_2\right\|_{C([0,\tau])}\left(\sum_{j=1}^{s}\|\tilde\psi_j'\|_{C([0,\tau])}\right). \tag{31}$$

The final estimate is of the form (see (25))

$$\left\|R(\vec\mu_1)-R(\vec\mu_2)\right\|_{C([0,\tau])}\le\|\mu_{1k}-\mu_{2k}\|_{C([0,\tau])}c_{15}\left(\tau^{\gamma}+\sum_{j=1}^{s}\|\tilde\psi_j'\|_{C([0,\tau])}\right). \tag{32}$$

Choosing $\tau_2\le\tau_1$ such that $c_{15}\left(\tau^{\gamma}+\sum_{j=1}^{s}\|\tilde\psi_j'\|_{C([0,\tau])}\right)\le\frac{1}{2}$ for $\tau\le\tau_2$, we have proven that R is a contraction and takes the ball B_{R_0} into itself for $\tau\le\tau_2$. The fixed point theorem implies the existence of a solution to the system (24). Let $v=v(\vec\mu)$. Show that this function satisfies the overdetemination conditions in (15). Take $x=b_j$ in (14). We obtain the system

$$c(t,b_j)v_t(t,b_j)+Av(t,b_j)=\sum_{i=1}^{r}\mu_iA_i(t,b_j,D)\Phi+\sum_{i=r+1}^{r_1}f_i\mu_i\Phi_t(t.b_j)+\sum_{j=r_1+1}^{S}f_j(t,b_j)\mu_j(t). \tag{33}$$

Subtracting these equalities from (21), we infer $v_t(t,b_j)-\psi_j'=0$ for all j and thereby these conditions are fulfilled. Uniqueness of a solution follows form the estimates exhibited above.

Remark 2. The corresponding stability estimate for solutions also holds.

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О НЕКОТОРЫХ КЛАССАХ ОБРАТНЫХ ПАРАБОЛИЧЕСКИХ ЗАДАЧ ВОССТАНОВЛЕНИЯ ТЕРМОФИЗИЧЕСКИХ ПАРАМЕТРОВ

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Аннотация. В работе мы рассматриваем вопросы регулярной разрешимости в соболевских пространствах обратных коэффициентных параболических задач. Решение ищется в классе регулярных решений, которые имеют все производные, входящие в уравнение суммируемые с некоторой степенью. В качестве условия переопределения берутся значения решения в некотором наборе точек, лежащих внутри области. Доказательство основано на априорных оценках и теореме о неподвижной точке.

Ключевые слова: параболическое уравнение; обратная задача; начально-краевая задача; существование, единственность.

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