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STOCHASTIC WENTZEL SYSTEM OF FREE FLUID FILTRATION EQUATIONS ON A HEMISPHERE AND ON ITS EDGE

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Abstract. Deterministic and stochastic Wentzell systems of the Dzekzer equations describing the evolution of the free surface of a filtering fluid in a hemisphere and at its edge are studied. In the deterministic case, the unambiguous solvability of the initial problem for the Wentzell system in a particular constructed Hilbert space is established. In the case of the stochastic system, the theory of Nelson–Glicklich derivatives is used and a stochastic solution is constructed to quantify the change in the free filtration of the fluid.

Keywords: stochastic Dzekzer equation; system of Wentzell equations; the Nelson–Glicklich derivative.

Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a manifold with an edge Γ. In particular, $\Omega = \{(\theta,\varphi) : \theta \in \left[0, \frac{\pi}{2}\right], \varphi \in [0, 2\pi]\}$ be a hemisphere in $\mathbb{R}^3$, and $\Gamma = \{\varphi : \varphi \in [0, 2\pi]\}$ } be a edge of hemisphere. The system of two Dzekzer equations [1], which describing free fluid filtration is defined on the compact $\Omega \cup \Gamma$

$$(\lambda - \Delta_{\theta,\varphi}) u_t = \alpha_0 \Delta_{\theta,\varphi} u - \beta_0 \Delta^2_{\theta,\varphi} u - \gamma_0 u, u = u(t,\theta,\varphi), (t,\theta,\varphi) \in \mathbb{R}_+ \times \Omega \quad (1)$$

$$(\lambda - \Delta_{\varphi}) v_t = \alpha_1 \Delta_{\varphi} u - \beta_1 \Delta^2_{\varphi} u + \partial_R u - \gamma_1 u, v = v(t,R,\varphi), (t,R,\varphi) \in \mathbb{R}_+ \times \Gamma, \quad (2)$$

where the Laplace–Beltrami operator $\Delta_{\theta,\varphi}$ on the hemisphere and the Laplace–Beltrami operator Δ_{φ} on the edge of the hemisphere have the following form

$$\Delta_{\theta,\varphi} = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2},\ \Delta_{\varphi} = \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2}, \partial_R = \left.\frac{\partial}{\partial\theta}\right|_{\theta=\frac{\pi}{2}} \quad (3)$$

Here, the symbol $\nu = \nu(t,\theta,\varphi), (t,\theta,\varphi) \in \mathbb{R}_+ \times \Gamma$, denotes the external normal to $\mathbb{R}_+ \times \Omega$. The parameters $\alpha_0, \alpha_1, \lambda, \beta_0, \beta_1, \gamma_0, \gamma_1 \in \mathbb{R}$ characterize the medium. To this system we add the matching condition

$$\operatorname{tr} u = v \text{ на } \mathbb{R}_+ \times \Gamma, \quad (4)$$

and equip it will initial conditions

$$u(0,\theta,\varphi) = u_0(\theta,\varphi), v(0,\varphi) = v_0(\varphi). \quad (5)$$

Let us call the solution of the problem (1)–(5) the deterministic solution of the Wentzell system. If we replace u and v, defined by Ω and Γ respectively, on $\eta = \eta(t)$ and $\kappa = \kappa(t)$ are stochastic processes on the interval $(0,\tau)$, we obtain stochastic Wentzell system, where the derivative of stochastic processes is understand by the Nelson–Gliklikh derivative of the process. It associated with correct definition of "white noise" as one-dimensional Wiener process (see, for example, [2–7]). Let us call the solution of the corresponding problem the stochastic solution of the Wentzell system.

The paper, in addition to the introduction and the list of references, consists of two parts. The first part considers the existence and uniqueness of the deterministic Wentzell system of equations of free filtration of fluid on a hemisphere and at its edge. The second part contains the proof of existence and uniqueness of the stochastic system of Wentzell equations of free fluid filtration on a hemisphere and at its edge.

The deterministic Wentzell system of free fluid filtration equations

If $\theta_k = k(k+1)$ eigenvalues of the Laplace–Beltrami operator $\Delta_{\theta,\varphi}$, then

$$Y_k^m(\varphi,\theta)=\begin{cases} P_k^m(\cos\theta)\cos m\varphi,\ m=0,...,k; \\ P_k^{|m|}(\cos\theta)\sin|m|\varphi,\ m=-k,...,-1, \end{cases}$$

are the corresponding eigenfunctions orthonormalized with respect to the scalar product. Here,

$$P_k(t)=\frac{1}{2^k k!}\frac{d^k}{dt^k}(t^2-1)^k$$

is a Lejandre polynomial of degree k, and $P_k^{|m|}(t)=\left(1-t^2\right)^{|m|/2}\frac{d^{|m|}}{dt^{|m|}}P_k(t)$ is the attached Lejandre polynomial. The scalar product is calculated using the following formula

$$\left\langle Y_{k_1}^{m_1},Y_{k_2}^{m_2}\right\rangle=\int_0^{2\pi}\cos m_1\varphi\cos m_2\varphi d\varphi\int_{-1}^{1}P_{k_1}^{m_1}(t)P_{k_2}^{m_2}(t)dt.$$

Consider the following series

$$u=\sum_{k=1}^{\infty}\sum_{m=0}^{k}\exp\left(t\frac{\beta_0k^4-\alpha_0k^2-\gamma_0}{\lambda+k^2}\right)\left(a_{m,k}\cos m\varphi+b_{m,k}\sin m\varphi\right)P_k^m(\cos\theta),\tag{6}$$

where

$$a_{m,k}=\int_0^{2\pi}u_0(\theta,\varphi)\cos m\varphi d\varphi\int_0^{\pi/2}P_k^m(0)\sin\theta d\theta,\ b_{m,k}=\int_0^{2\pi}u_0(\theta,\varphi)\sin m\varphi d\varphi\int_0^{\pi/2}P_k^m(0)\sin\theta d\theta.$$

It is easy to see that the series constructed above is a formal solution of the equation (1). Moreover, if the series in (6) converge uniformly, then we have a solution to the problem (1), (5), where $\partial_\theta u=0$. Given this, we can construct a solution to the problem (2), (5)

$$u=\sum_{k=1}^{\infty}\exp\left(t\frac{\beta_1k^4-\alpha_1k^2-\gamma_1}{\lambda+k^2}\right)\left(c_k\cos k\varphi+d_k\sin k\varphi\right),\tag{7}$$

where

$$c_k=\int_0^{2\pi}v_0(\varphi)\cos k\varphi d\varphi,\ d_k=\int_0^{2\pi}v_0(\varphi)\sin k\varphi d\varphi.$$

In the case of the matching condition (4) we obtain the following equation

$$\sum_{k=1}^{\infty}\sum_{m=0}^{k}\exp\left(t\frac{\beta_0k^4-\alpha_0k^2-\gamma_0}{\lambda+k^2}\right)\left(a_{m,k}\cos m\varphi+b_{m,k}\sin m\varphi\right)P_k^m(\cos\theta)\Big|_{\theta=\pi/2}$$

$$=\sum_{k=1}^{\infty}\exp\left(t\frac{\beta_1k^4-\alpha_1k^2-\gamma_1}{\lambda+k^2}\right)\left(c_k\cos k\varphi+d_k\sin k\varphi\right).$$

Considering, that $\beta_0=\beta_1$, $\alpha_0=\alpha_1$, $\gamma_0=\gamma_1$ we obtain equalivent system of equations

$$\sum_{m=0}^{k}\left(a_{m,k}\cos m\varphi+b_{m,k}\sin m\varphi\right)P_k^m(0)=c_k\cos k\varphi+d_k\sin k\varphi,\text{ where } m+n=2k.$$

Substituting the integral coefficients we obtain an equivalent system

$$\sum_{m=0}^{k}\Big(\int_0^{2\pi}u_0(\theta,\varphi)\cos m\varphi d\varphi\int_0^{\pi/2}P_k^m(0)\sin\theta d\theta\cos m\varphi+\int_0^{2\pi}u_0(\theta,\varphi)\sin m\varphi d\varphi\int_0^{\pi/2}P_k^m(0)\sin\theta d\theta\sin m\varphi P_k^m(0)$$

$$=\int_0^{2\pi}v_0(\varphi)\cos k\varphi d\varphi\cos k\varphi+\int_0^{2\pi}v_0(\varphi)\sin k\varphi d\varphi\sin k\varphi.$$

Here the auxiliary integrals are calculated by the formula

$$\int_0^{\pi/2}P_k^m(0)\sin\theta d\theta=P_k^m(0)\int_0^{\pi/2}\sin\theta d\theta=P_k^m(0),$$

and system has the following form

$$\sum_{m=0}^{k}\left(\int_0^{2\pi} u_0(\theta,\varphi)\cos m\varphi d\varphi\cos m\varphi+\int_0^{2\pi} u_0(\theta,\varphi)\sin m\varphi d\varphi\sin m\varphi\right)(P_k^m(0))^2$$

$$=\int_0^{2\pi} v_0(\varphi)\cos k\varphi d\varphi\cos k\varphi+\int_0^{2\pi} v_0(\varphi)\sin k\varphi d\varphi\sin k\varphi. \quad (8)$$

Thus in the case $\beta_0=\beta_1$, $\alpha_0=\alpha_1$, $\gamma_0=\gamma_1$ and the obtained condition (8) the solutions to the problem (1)–(5) will satisfy the matching condition (4).

Lineal closure of the $\operatorname{span}\{P_k^m(\cos\theta)\sin m\varphi,\ P_k^m(\cos\theta)\cos m\varphi:$ $m,k\in\mathbb{N}\setminus\{1\},\theta\in\left[0,\frac{\pi}{2}\right],\varphi\in[0,2\pi)\}$ generated by the scalar product

$$\langle\varphi,\psi\rangle=\int_0^{2\pi}\int_0^{\pi/2}\varphi(\theta,\varphi)\psi(\theta,\varphi)\sin\theta d\theta d\varphi,$$

we denote by the symbol $A(\Omega)$. Next, the closure of the $\operatorname{span}\{\sin k\varphi,\ \cos k\varphi:\ k\in\mathbb{N},\varphi\in[0,2\pi)\}$ by the norm, generated by the scalar product

$$\langle\xi,\psi\rangle=\int_0^{2\pi}\xi(\varphi)\psi(\varphi)d\varphi,$$

we denote by the symbol $A(\Gamma)$.

Thus, the following theorem holds.

Theorem 2.1 *For any $u_0\in A(\Omega)$ and $v_0\in A(\Gamma)$, and any coefficients $\alpha_0,\alpha_1,\beta_0,\beta_1,\gamma_0,\gamma_1,\lambda\in\mathbb{R}$, such, that the conditions $\alpha_0=\alpha_1$, $\beta_0=\beta_1$, $\gamma_0=\gamma_1$, $\lambda\neq k^2$ are satisfied, where $k\in\mathbb{N}$, and the system (8) is solvable, then there exists a unique solution $(u,v)\in C^\infty(\mathbb{R};A(\Omega)+A(\Gamma))$ of problem (1)–(5).*

The stochastic Wentzell system of free fluid filtration equations

For simplicity's sake, let $\mathfrak{A}=\{u\ \in W_2^2(\Omega)+W_2^2(\Gamma):\partial_R u=0\}$, $\mathfrak{F}=L_2(\Omega)+L_2(\Gamma)$. Next, following the algorithm above, construct the spaces of random K-values. The random K-values $\eta,\kappa\in U_K L_2$ has the form $\eta=\sum_{i=1}^{\infty}\lambda_i\eta_i\varphi_i$, $\kappa=\sum_{k=1}^{\infty}\lambda_k\kappa_k\psi_k$, where $\{\varphi_k\}$ is the family of eigenfunctions of the Laplace – Beltrami operator $\Delta_{\theta,\varphi}\in L(U_K L_2;F_K L_2)$ orthonormalized in the sense of the scalar product $(\cdot,\cdot)$ of $L_2(\Omega)$; $\{\psi_k\}$ is the family of eigenfunctions of the Laplace–Beltrami operator $\Delta_\varphi\in L(U_K L_2;F_K L_2)$ orthonormalized in the sense of the scalar product $(\cdot,\cdot)$ of $L_2(\Omega)$. Consider the linear stochastic Wentzel system of free fluid filtration equations in a hemisphere and at its edge. In this case (1)–(5) is transformed to the form

$$(\lambda-\Delta_{\theta,\varphi})\eta_t=\alpha_0\Delta_{\theta,\varphi}\eta-\beta_0\Delta_{\theta,\varphi}^2\eta-\gamma_0\eta,\ \eta\in C^\infty(\mathbb{R}_+;U_K L_2), \quad (9)$$

$$(\lambda-\Delta_\varphi)\kappa_t=\alpha_1\Delta_\varphi\kappa-\beta_1\Delta_\varphi^2\kappa+\partial_R\eta-\gamma_1\kappa,\ \kappa\in C^\infty(\mathbb{R}_+;U_K L_2) \quad (10)$$

To the system (9), (10) we add the corresponding matching condition (8) and initial condition

$$\eta(0)=\eta_0,\kappa(0)=\kappa_0, \quad (11)$$

The solution of the problem (9)–(11) will be called a stochastic solution. Thus, using the idea inherent in the results obtained earlier (see, for example [8]), the following theorem holds.

Theorem 3.1 *For any $\eta_0,\kappa_0\in U_K L_2(\Omega)$ and any coefficients $\alpha_0,\alpha_1,\beta_0,\beta_1,\gamma_0,\gamma_1,\lambda\in\mathbb{R}$, such, that the conditions $\alpha_0=\alpha_1$, $\beta_0=\beta_1$, $\gamma_0=\gamma_1$, $\lambda\neq k^2$ are satisfied, where $k\in\mathbb{N}$, and the system (8) is solvable, then there exists a unique solution $\eta\in C^\infty(\mathbb{R}_+;U_K L_2)$ of problems (9)–(11).*

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СТОХАСТИЧЕСКАЯ СИСТЕМА ВЕНТЦЕЛЯ УРАВНЕНИЙ СВОБОДНОЙ ФИЛЬТРАЦИИ ЖИДКОСТИ НА ПОЛУСФЕРЕ И НА ЕЕ КРАЕ

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Аннотация. Исследуются детерминированные и стохастические системы Вентцеля уравнений Дзекцера, описывающие эволюцию свободной поверхности фильтрующейся жидкости на полусфере и на ее краю. В детерминированном случае установлена однозначная разрешимость начальной задачи для системы Вентцеля в конкретном построенном гильбертовом пространстве. В случае стохастической системы используется теория производных Нельсона–Гликлиха и строится стохастическое решение, позволяющее определить количественное изменение свободной фильтрации жидкости.

Ключевые слова: стохастическое уравнение Дзекцера; система уравнений Вентцеля; производная Нельсона–Гликлиха.

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