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ANALYSIS OF A SCREENED HARMONIC SYSTEM UNDER DIRICHLET AND NEUMANN BOUNDARY CONDITIONS

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Abstract. In this paper, a screened harmonic system with Dirichlet and Neumann boundary conditions in a domain with complex geometry is considered, and a method for analyzing such a system is proposed. The development of this method is especially relevant for solving boundary value problems for the screened Poisson equation in domains with complex geometry, which are used to describe various physical systems in mechanics, hydrodynamics, electrical engineering, and heat engineering. The proposed algorithm for analyzing a screened harmonic system under these boundary conditions makes a significant contribution to this area. The proposed method includes the continuation of the screened harmonic system through boundaries with Dirichlet and Neumann conditions. Then, the continuation is discretized by a system of linear algebraic equations. An asymptotically optimal analysis of the discrete continued screened harmonic system and an algorithm implementing the method for analyzing the screened harmonic system with optimal asymptotics in the number of arithmetic operations are carried out.

Keywords: Screened harmonic system; asymptotically optimal analysis.

Introduction

The development of an asymptotically optimal method for solving a boundary value problem for the screened Poisson equation is relevant for the analysis of screened harmonic systems in domains with complex geometry, which describe the corresponding physical systems. For the screened Poisson equation, a boundary value problem in a domain with complex geometry is used to describe a stationary physical system in nature and technology, for example, in mechanics, hydrodynamics, electrical engineering, heat engineering, etc. Such a problem was studied within the framework of similar approaches, for example, in works [1, 2]. While creating numerical techniques for the analysis of screened harmonic systems in domains with complex geometry, they are reduced to a system in rectangular domains for which asymptotically optimal marching methods are known [3]. In order to achieve results with optimal asymptotics for elliptic problems with the Neumann boundary condition, a methodology of fictitious components for solving second-order elliptic boundary value problems in the presence of a Dirichlet boundary condition was proposed, studied, and optimized in the works [4–6]. This paper is devoted to the numerical solution of a mixed boundary value problem, which describes the displacements of membrane points under transverse pressure with fixed and free edges based on its analysis as a screened harmonic system:

$$u: -\Delta\breve{u} + \kappa\breve{u} = \breve{f}\,|_{\Omega},\ \Omega \subset R^2,\ \kappa \ge 0, \tag{1}$$

$$\breve{u}\Big|_{\Gamma_1} = 0,\ \frac{\partial\breve{u}}{\partial\vec{n}}\Big|_{\Gamma_2} = 0,$$

where

$$\partial\Omega = \overline{s},\ s = \Gamma_1 \bigcup \Gamma_2,\ \Gamma_1 \bigcap \Gamma_2 = \varnothing.$$

1. Screened harmonic system and its continuation

For the screened Poisson equation, the boundary value problem is considered as a screened harmonic system in two versions with variable index $\omega \in W$, $W = \{1, \mathrm{II}\}$. When $\omega \in W$ we consider the problem (1), and when $W \setminus w$ we introduce the fictitious problem. From the theory of elasticity comes definition of energy of a deformed membrane:

$$\breve{E}_\omega(\breve{u}_\omega) = \frac{1}{2}\breve{T}_\omega \int\limits_{\Omega_\omega} (\breve{u}_{\omega x}^2 + \breve{u}_{\omega y}^2) d\Omega_\omega + \frac{1}{2}\int\limits_{\Omega_\omega} \breve{K}_\omega \breve{u}_\omega^2 d\Omega_\omega - \int\limits_{\Omega_\omega} \breve{P}_\omega \breve{u}_\omega d\Omega_\omega,$$

where the displacement of the membrane is $\breve{u}_\omega$, the membrane tension coefficient is $\breve{T}_\omega > 0$, the elastic foundation stiffness coefficient is $\breve{K}_\omega \geq 0$, the pressure is $\breve{P}_\omega$, a limited domain Ω_ω with a piecewise smooth boundary $\partial\Omega_\omega$ of class C^2 without self-tangencies and self-intersections, $\partial\Omega_\omega = \bar{s}_\omega$, $s_\omega = \Gamma_{\omega,1} \bigcup \Gamma_{\omega,2}$, $\Gamma_{\omega,i} \bigcap \Gamma_{\omega,j} = \varnothing$, if $i \neq j$, $i, j = 1,2.$ The variation of energy of membrane equates to zero

$$\delta\breve{E}_\omega(\breve{u}_\omega) = \breve{T}_\omega \int\limits_{\Omega_\omega} (\breve{u}_{\omega x}\breve{v}_{\omega x} + \breve{u}_{\omega y}\breve{v}_{\omega y}) d\Omega_\omega + \int\limits_{\Omega_\omega} \breve{K}_\omega \breve{u}_\omega \breve{v}_\omega d\Omega_\omega - \int\limits_{\Omega_\omega} \breve{P}_\omega \breve{v}_\omega d\Omega_\omega = 0,$$

if $\breve{v}_\omega = \delta\breve{u}_\omega$, $\kappa_\omega = \breve{K}_\omega / \breve{T}_\omega$, $\breve{f}_\omega = \breve{P}_\omega / \breve{T}_\omega$, then

$$\int\limits_{\Omega_\omega} (\breve{u}_{\omega x}\breve{v}_{\omega x} + \breve{u}_{\omega y}\breve{v}_{\omega y} + \kappa_\omega \breve{u}_\omega \breve{v}_\omega) d\Omega_\omega = \int\limits_{\Omega_\omega} \breve{f}_\omega \breve{v}_\omega d\Omega_\omega.$$

We integrate by parts

$$\int\limits_{\Omega_\omega} (-\Delta\breve{u}_\omega + \kappa_\omega \breve{u}_\omega)\breve{v}_\omega d\Omega_\omega + \int\limits_{s_\omega} \frac{\partial \breve{u}_\omega}{\partial n_\omega} \breve{v}_\omega ds_\omega = \int\limits_{\Omega_\omega} \breve{f}_\omega \breve{v}_\omega d\Omega_\omega.$$

If $\vec{n}_\omega$ is outer normal to $\partial\Omega_\omega$, the membrane is fixed on $\Gamma_{\omega,1}$, and the membrane is free on $\Gamma_{\omega,2}$, then boundary value problem arises:

$$-\Delta\breve{u}_\omega + \kappa_\omega \breve{u}_\omega = \breve{f}_\omega \Big|_{\Omega_\omega}, \ \kappa_\omega \geq 0,$$

$$\breve{u}_\omega \Big|_{\Gamma_{\omega,1}} = 0, \ \frac{\partial \breve{u}_\omega}{\partial \vec{n}_\omega} \Big|_{\Gamma_{\omega,2}} = 0.$$

Same boundary value problem in variational form

$$\breve{u}_\omega \in \breve{H}_\omega : \mathrm{A}_\omega(\breve{u}_\omega, \breve{v}_\omega) = F_\omega(\breve{v}_\omega) \ \forall \breve{v}_\omega \in \breve{H}_\omega, \ F_\omega \in \breve{H}'_\omega \tag{2}$$

in the space of Sobolev functions

$$\breve{H}_\omega = \breve{H}_\omega(\Omega_\omega) = \left\{ \breve{v}_\omega \in W_2^1(\Omega_\omega) : \breve{v}_\omega \big|_{\Gamma_{\omega,1}} = 0 \right\}.$$

The dot product in bilinear form is defined as

$$\mathrm{A}_\omega(\breve{u}_\omega, \breve{v}_\omega) = \int\limits_{\Omega_\omega} (\breve{u}_{\omega x}\breve{v}_{\omega x} + \breve{u}_{\omega y}\breve{v}_{\omega y} + \kappa_\omega \breve{u}_\omega \breve{v}_\omega) d\Omega_\omega.$$

The assumption below implies that the problem's solution exists and is unique

$$\exists c_1, c_2 \in (0; +\infty) : c_1 \left\| \breve{v}_\omega \right\|^2_{W_2^1(\Omega_\omega)} \leq \mathrm{A}_\omega(\breve{v}_\omega, \breve{v}_\omega) \leq c_2 \left\| \breve{v}_\omega \right\|^2_{W_2^1(\Omega_\omega)} \ \forall \breve{v}_\omega \in \breve{H}_\omega,$$

if for a given function $\breve{f}_\omega \in L_2(\Omega_\omega)$ the linear functional is

$$F_\omega(\breve{v}_\omega) = \int\limits_{\Omega_\omega} \breve{f}_\omega \breve{v}_\omega d\Omega_\omega.$$

Let us consider fictitious problem

$$-\Delta\breve{u}_{W\backslash\omega} + \kappa_{W\backslash\omega} \breve{u}_{W\backslash\omega} = \breve{f}_{W\backslash\omega}, \ \breve{f}_{W\backslash\omega} = 0,$$

$$\breve{u}_{W\backslash\omega} \Big|_{\Gamma_{W\backslash\omega,1}} = 0, \ \frac{\partial \breve{u}_{W\backslash\omega}}{\partial \vec{n}_{W\backslash\omega}} \Big|_{\Gamma_{W\backslash\omega,2}} = 0$$

Same problem in variational form:

$$\breve{u}_{W\backslash\omega} \in \breve{H}_{W\backslash\omega} : \mathrm{A}_{W\backslash\omega}(\breve{u}_{W\backslash\omega}, \breve{v}_{W\backslash\omega}) = F_{W\backslash\omega}(\breve{v}_\omega) \ \forall \breve{v}_{W\backslash\omega} \in \breve{H}_{W\backslash\omega}, \ F_{W\backslash\omega}(\breve{v}_\omega) = 0.$$

A continuation of the boundary value problem is performed for the screened Poisson equation, which is regarded as an extension of the screened harmonic system:

$$\breve{u} \in \breve{V} : \ \mathrm{A}_1(\breve{u}, I_1\breve{v}) + \mathrm{A}_{\mathrm{II}}(\breve{u}, \breve{v}) = F_1(I_1\breve{v}) + F_{\mathrm{II}}(\breve{v}) \forall \breve{v} \in \breve{V} \tag{3}$$

on extended space

$$\breve{V} = \breve{V}(\Pi) = \left\{ \breve{v} \in W_2^1(\Pi) : \breve{v}\big|_{\Gamma_1} = 0 \right\},$$

assuming, that domains are

$$\bar{\Omega}_1 \bigcup \bar{\Omega}_{\mathrm{II}} = \bar{\Pi},\ \Omega_1 \bigcap \Omega_{\mathrm{II}} = \varnothing,\ \Omega_1, \Omega_{\mathrm{II}} \subset R^2,$$

and the boundary of the domain Π is a piecewise smooth boundary of class C^2 without self-tangencies and self-intersections

$$\partial\Pi = \bar{s},\ s = \Gamma_1 \bigcup \Gamma_2,\ \Gamma_i \bigcap \Gamma_j = \varnothing,\ i \neq j,\ i, j = 1,2$$

At the same time

$$\partial\Omega_1 \bigcap \partial\Omega_{\mathrm{II}} = \bar{S},\ S = \Gamma_{1,1} \bigcap \Gamma_{\mathrm{II},2} \neq \varnothing.$$

The solution space of the extended problem $\breve{V}_1$ is the subspace of the extended space

$$\breve{V}_1 = \breve{V}_1(\Pi) = \left\{ \breve{v}_1 \in \breve{V} : \breve{v}_1\big|_{\Pi \setminus \Omega_1} = 0 \right\}.$$

We use arbitrary projection operators

$$I_1 : \breve{V} \mapsto \breve{V}_1,\ \breve{V}_1 = imI_1,\ I_1 = I_1^2.$$

Additionally, we define subspaces of the extended space

$$\breve{V}_3 = \breve{V}_3(\Pi) = \left\{ \breve{v}_3 \in \breve{V} : \breve{v}_3\big|_{\Pi \setminus \Omega_{\mathrm{II}}} = 0 \right\},\ \breve{V}_0 = \breve{V}_1 \oplus \breve{V}_3,$$

$$\breve{V}_2 = \breve{V}_2(\Pi) = \left\{ \breve{v}_2 \in \breve{V} : \mathrm{A}(\breve{v}_2, \breve{v}_0) = 0\ \forall \breve{v}_0 \in \breve{V}_0 \right\},$$

$$\breve{V} = \breve{V}_1 \oplus \breve{V}_2 \oplus \breve{V}_3 = \breve{V}_1 \oplus \breve{V}_{\mathrm{II}},\ \breve{V}_{\mathrm{I}} = \breve{V}_1 \oplus \breve{V}_2,\ \breve{V}_{\mathrm{II}} = \breve{V}_2 \oplus \breve{V}_3.$$

The decomposition into direct sums here is determined by the the bilinear form

$$\mathrm{A}(\breve{u}, \breve{v}) = \mathrm{A}_1(\breve{u}, \breve{v}) + \mathrm{A}_{\mathrm{II}}(\breve{u}, \breve{v})\ \forall \breve{u}, \breve{v} \in \breve{V}.$$

We consider bilinear form to be such that following inequality takes place:

$$\exists c_1, c_2 > 0 : c_1 \|\breve{v}\|^2_{W_2^1(\Pi)} \le \mathrm{A}(\breve{v}, \breve{v}) \le c_2 \|\breve{v}\|^2_{W_2^1(\Pi)}\ \forall \breve{v} \in \breve{V},$$

and continuation of functions with preservation of the norm and class take place in the following form:

$$\exists \breve{\beta}_1 \in (0;1],\ \breve{\beta}_2 \in [\breve{\beta}_1;1] : \breve{\beta}_1 \mathrm{A}(\breve{v}_2, \breve{v}_2) \le \mathrm{A}_{\mathrm{II}}(\breve{v}_2, \breve{v}_2) \le \breve{\beta}_2 \mathrm{A}(\breve{v}_2, \breve{v}_2)\ \forall \breve{v}_2 \in \breve{V}_2.$$

Let us denote solution to the problem (2) and solution to the continued problem (3) as function in the same way, and

$$\breve{H}_\omega(\Omega_\omega) = \breve{V}_\omega(\Omega_\omega),\ \omega \in \{1, \mathrm{II}\}.$$

Proposition 1. The following dot products are equal to zero:

$$\mathrm{A}_\omega(\breve{u}_0, \breve{v}_2) = \mathrm{A}_\omega(\breve{v}_2, \breve{u}_0) = 0\ \ \forall \breve{u}_0 \in \breve{V}_0,\ \forall \breve{v}_2 \in \breve{V}_2,\ \ \omega \in \{1, \mathrm{II}\}.$$

Proof. This is obtained from the following equalities

$$\mathrm{A}_1(\breve{u}_0, \breve{v}_2) = \mathrm{A}_1(\breve{u}_1, \breve{v}_2) = \mathrm{A}(\breve{u}_1, \breve{v}_2) = 0\ \ \forall \breve{u}_1 \in \breve{V}_0,\ \forall \breve{v}_2 \in \breve{V}_2,$$

$$\mathrm{A}_{\mathrm{II}}(\breve{u}_0, \breve{v}_2) = \mathrm{A}_{\mathrm{II}}(\breve{u}_3, \breve{v}_2) = \mathrm{A}(\breve{u}_3, \breve{v}_2) = 0\ \ \forall \breve{u}_3 \in \breve{V}_0,\ \forall \breve{v}_2 \in \breve{V}_2.$$

Statement 1. The solution to problem (3) $\breve{u} \in \breve{V}_\omega$ exists, is unique, matches with solution to the problem (2) on Ω_ω, and this is $\breve{u}_2 \in \breve{V}_2$ (zero at $\omega = 1$) on $\Omega_{W \setminus \omega}$.

Proof. If $\breve{u}^1$, $\breve{u}^2$ are both solutions, and $\breve{u}^0 = \breve{u}^1 - \breve{u}^2$, then

$$\mathrm{A}_1(\breve{u}^0, I_1\breve{v}) + \mathrm{A}_{\mathrm{II}}(\breve{u}^0, \breve{v}) = 0\ \ \forall \breve{v} \in \breve{V}.$$

Then $\breve{u}^0 = \breve{u}_2^0 \in \breve{V}_2$, because for $\breve{v} = \breve{v}_0$ we have

$$\mathrm{A}_1(\breve{u}^0, I_1\breve{v}_0) + \mathrm{A}_{\mathrm{II}}(\breve{u}^0, \breve{v}_0) = \mathrm{A}_1(\breve{u}^0, \breve{v}_0) + \mathrm{A}_{\mathrm{II}}(\breve{u}^0, \breve{v}_0) = \mathrm{A}(\breve{u}^0, \breve{v}_0) = 0\ \ \forall \breve{v}_0 \in \breve{V}_0.$$

If $\breve{v} = \breve{u}_2^0$, then

$$\mathrm{A}_1(\breve{u}_2^0, I_1\breve{u}_2^0) + \mathrm{A}_{\mathrm{II}}(\breve{u}_2^0, \breve{u}_2^0) = 0,$$

and according to proposition 2.1 we have

$$\mathrm{A}_1(\breve{u}_2^0, I_1\breve{u}_2^0) = 0, \mathrm{A}_{\mathrm{II}}(\breve{u}_2^0, \breve{u}_2^0) = 0,$$

and $\breve{u}_2^0 = 0$ on Ω_{II}, because $\mathrm{A}_{\mathrm{II}}(.\,,.)$ is dot product, and

$$0 \le \breve{\beta}_1 \mathrm{A}(u_2^0, u_2^0) \le \mathrm{A}_{\mathrm{II}}(u_2^0, u_2^0) = 0,$$

then

$$A(u_2^0, u_2^0) = 0,$$

and $\breve{u}_2^0 = 0$ on Π, because $A(.,.)$ is dot product. Therefore, $\breve{u}^0 = \breve{u}_2^0 = 0$ on Π. The existing solution to the problems in (3) is indicated in the formulation of the statement.

2. Continued system in finite-dimensional subspace

Let us consider the discretization of the continued problem with

$$\Pi = (0;b_1)\times(0;b_2),\ \Gamma_1 = \{b_1\}\times(0;b_2)\cup(0;b_1)\times\{b_2\},$$

$$\Gamma_2 = \{0\}\times(0;b_2)\cup(0;b_1)\times\{0\},\ b_1, b_2 \in (0;+\infty).$$

In area Π we define the grid nodes

$$(x_i; y_j) = \left((i-1{,}5)h_1; (j-1{,}5)h_2\right),$$

$$h_1 = b_1/(m-1{,}5),\ h_2 = b_2/(n-1{,}5),\ i = 1,2...,m,\ j = 1,2...,n,\ m-2, n-2 \in \mathrm{N}.$$

We define grid functions on grid nodes

$$v_{i,j} = v(x_i; y_j) \in R,\ i = 1,2...,m,\ j = 1,2...,n,\ m-2, n-2 \in \mathrm{N}.$$

We take into account boundary conditions and define linear basis

$$\Phi^{i,j}(x;y) = \Psi^{1,i}(x)\Psi^{2,j}(y),\ i = 2...,m-1,\ j = 2...,n-1,\ m-2, n-2 \in \mathrm{N},$$

$$\Psi^{1,i}(x) = [2/i]\Psi(x/h_1 - i + 3{,}5) + \Psi(x/h_1 - i + 2{,}5),$$

$$\Psi^{2,j}(y) = [2/j]\Psi(y/h_2 - j + 3{,}5) + \Psi(y/h_2 - j + 2{,}5),$$

$$\Psi(z) = \begin{cases} z, & z \in [0;1], \\ 2-z, & z \in [1;2], \\ 0, & z \notin (0;2), \end{cases}$$

where function $[\,.\,]$ is integer part of a number. We additionally determine that

$$\Phi^{i,j}(x;y) = 0,\ (x;y) \notin \Pi,\ i = 2...,m-1,\ j = 2...,n-1,\ m-2, n-2 \in \mathrm{N}.$$

We will use the approximation of the extended space by a finite-dimensional subspace

$$\tilde{V} = \left\{ \tilde{v} = \sum_{i=2}^{m-1}\sum_{j=2}^{n-1} v_{i,j}\Phi^{i,j}(x;y) \right\} \subset \breve{V}.$$

Let us consider the continued problem in a finite-dimensional subspace

$$\tilde{u} \in \tilde{V}:\ A_1(\tilde{u}, I_1\tilde{v}) + A_{\mathrm{II}}(\tilde{u}, \tilde{v}) = F_1(I_1\tilde{v}) + F_{\mathrm{II}}(\tilde{v})\ \forall \tilde{v} \in \tilde{V}.$$

Finite-dimensional subspace of solutions of the continued problem

$$\tilde{V}_1 = \tilde{V}_1(\Pi) = \left\{ \tilde{v}_1 \in \tilde{V} : \tilde{v}_1\big|_{\Pi\setminus\Omega_1} = 0 \right\}.$$

We assume that the projection operator

$$I_1 : \tilde{V} \mapsto \tilde{V}_1,\ \tilde{V}_1 = im I_1,\ I_1 = I_1^2$$

determined as

$$I_1\left(\sum_{i=2}^{m-1}\sum_{j=2}^{n-1} v_{i,j}\Phi^{i,j}(x;y) \right) = \sum_{i=2}^{m-1}\sum_{j=2}^{n-1} v_{i,j} I_1\left(\Phi^{i,j}(x;y)\right),$$

$$I_1\left(\Phi^{i,j}(x;y)\right) = \begin{cases} \Phi^{i,j}(x;y), & \mathrm{supp}\left\{\Phi^{i,j}\right\} \subset \bar{\Omega}_1, \\ 0, & \mathrm{supp}\left\{\Phi^{i,j}\right\} \not\subset \bar{\Omega}_1. \end{cases}$$

We assume that, for example, for a function f set $\mathrm{supp}\{f\}$ denotes its support. Let us define subspaces in a finite-dimensional subspace

$$\tilde{V}_3 = \tilde{V}_3(\Pi) = \left\{ \tilde{v}_3 \in \tilde{V} : \tilde{v}_3\big|_{\Pi\setminus\Omega_{\mathrm{II}}} = 0 \right\},\ \tilde{V}_0 = \tilde{V}_1 \oplus \tilde{V}_3,$$

$$\tilde{V}_2 = \tilde{V}_2(\Pi) = \left\{ \tilde{v}_2 \in \tilde{V} : A(\tilde{v}_2, \tilde{v}_0) = 0\ \forall \tilde{v}_0 \in \tilde{V}_0 \right\},$$

$$\tilde{V} = \tilde{V}_1 \oplus \tilde{V}_2 \oplus \tilde{V}_3 = \tilde{V}_1 \oplus \tilde{V}_{\mathrm{II}},\ \tilde{V}_{\mathrm{I}} = \tilde{V}_1 \oplus \tilde{V}_2,\ \tilde{V}_{\mathrm{II}} = \tilde{V}_2 \oplus \tilde{V}_3.$$

We assume that the provisions on the possibility of continuing functions with preservation of the norm and class on a finite-dimensional space are fulfilled in the following form:

$$\exists \tilde{\beta}_1 \in (0;1],\ \tilde{\beta}_2 \in [\tilde{\beta}_1;1]:\ \tilde{\beta}_1 \mathrm{A}(\tilde{v}_2,\tilde{v}_2) \le \mathrm{A}_{\mathrm{II}}(\tilde{v}_2,\tilde{v}_2) \le \tilde{\beta}_2 \mathrm{A}(\tilde{v}_2,\tilde{v}_2)\ \forall \tilde{v}_2 \in \tilde{V}_2.$$

3. Continued system on Euclidean space

We approximate the continued problem and obtain a system of equations

$$\bar{u} \in R^N:\ B\bar{u} = \bar{f},\ \bar{f} \in R^N, \tag{4}$$

where matrix B and right side $\bar{f}$ are defined as follows

$$\langle B\bar{u},\bar{v}\rangle = \mathrm{A}_1(\tilde{u}, I_1\tilde{v}) + \mathrm{A}_{\mathrm{II}}(\tilde{u},\tilde{v})\ \forall \tilde{u},\tilde{v} \in \tilde{V},\ \langle \bar{f},\bar{v}\rangle = F_1(I_1\tilde{v})\ \forall \tilde{v} \in \tilde{V},$$

$$\langle \bar{f},\bar{v}\rangle = (\bar{f},\bar{v})h_1h_2 = \bar{f}\,\bar{v}h_1h_2,\ \bar{v} = (v_1, v_2, \ldots, v_N)' \in R^N,\ N = (m-2)(n-2).$$

In this system of equations, we enumerate first the basis functions $\Phi^{i,j}$ and their coefficients, if

$$\operatorname{supp}\left\{\Phi^{i,j}\right\} \subset \bar{\Omega}_1,$$

then enumerate second the basis functions $\Phi^{i,j}$ and their coefficients, if

$$\operatorname{supp}\left\{\Phi^{i,j}\right\} \cap \Omega_1 \ne \varnothing \wedge \operatorname{supp}\left\{\Phi^{i,j}\right\} \cap \Omega_{\mathrm{II}} \ne \varnothing,$$

and enumerate third the basis functions $\Phi^{i,j}$ and their coefficients, if

$$\operatorname{supp}\left\{\Phi^{i,j}\right\} \subset \bar{\Omega}_{\mathrm{II}}.$$

With this enumeration, the vectors are

$$\bar{v} = (\bar{v}_1', \bar{v}_2', \bar{v}_3')',\ \bar{u} = (\bar{u}_1', \bar{u}_2', \bar{u}_3')',\ \bar{f} = (\bar{f}_1', \bar{f}_2', \bar{f}_3')'.$$

The matrix B is obtained in block form

$$B = \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} & 0 \\ 0 & \mathrm{A}_{02} & \mathrm{A}_{23} \\ 0 & \mathrm{A}_{32} & \mathrm{A}_{33} \end{bmatrix}.$$

Additionally, we define the matrices A_{I}, A_{II}:

$$\langle \mathrm{A}_{\mathrm{I}}\bar{u},\bar{v}\rangle = \mathrm{A}_1(\tilde{u},\tilde{v}),\ \langle \mathrm{A}_{\mathrm{II}}\bar{u},\bar{v}\rangle = \mathrm{A}_{\mathrm{II}}(\tilde{u},\tilde{v})\ \forall \tilde{u},\tilde{v} \in \tilde{V}.$$

The matrices A_{I}, A_{II} are obtained in a block form

$$\mathrm{A}_{\mathrm{I}} = \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} & 0 \\ \mathrm{A}_{21} & \mathrm{A}_{20} & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \mathrm{A}_{\mathrm{II}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathrm{A}_{02} & \mathrm{A}_{23} \\ 0 & \mathrm{A}_{32} & \mathrm{A}_{33} \end{bmatrix}.$$

We introduce an extended matrix

$$\mathrm{A} = \mathrm{A}_{\mathrm{I}} + \mathrm{A}_{\mathrm{II}} = \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} & 0 \\ \mathrm{A}_{21} & \mathrm{A}_{22} & \mathrm{A}_{23} \\ 0 & \mathrm{A}_{32} & \mathrm{A}_{33} \end{bmatrix} = \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} & 0 \\ \mathrm{A}_{21} & \mathrm{A}_{20} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathrm{A}_{02} & \mathrm{A}_{23} \\ 0 & \mathrm{A}_{32} & \mathrm{A}_{33} \end{bmatrix}.$$

Let us define vector subspaces

$$\bar{V}_1 = \left\{\bar{v} = (\bar{v}_1', \bar{v}_2', \bar{v}_3')' \in R^N : \bar{v}_2 = \bar{0},\ \bar{v}_3 = \bar{0}\right\},$$

$$\bar{V}_3 = \left\{\bar{v} = (\bar{v}_1', \bar{v}_2', \bar{v}_3')' \in R^N : \bar{v}_1 = \bar{0},\ \bar{v}_2 = \bar{0}\right\},\ \bar{V}_0 = \bar{V}_1 \oplus \bar{V}_3,$$

$$\bar{V}_2 = \left\{\bar{v} = (\bar{v}_1', \bar{v}_2', \bar{v}_3')' \in R^N : \mathrm{A}_{11}\bar{v}_1 + \mathrm{A}_{12}\bar{v}_2 = \bar{0},\ \mathrm{A}_{32}\bar{v}_2 + \mathrm{A}_{33}\bar{v}_3 = \bar{0}\right\}.$$

We note that

$$R^N = \bar{V}_1 \oplus \bar{V}_2 \oplus \bar{V}_3 = \bar{V}_1 \oplus \bar{V}_{\mathrm{II}},\ \bar{V}_{\mathrm{I}} = \bar{V}_1 \oplus \bar{V}_2,\ \bar{V}_{\mathrm{II}} = \bar{V}_2 \oplus \bar{V}_3.$$

We assume that the provisions on the possibility of continuing functions with preservation of the norm and class on a finite-dimensional space are fulfilled in the following form:

$$\exists\beta_1 \in (0;+\infty),\ \beta_2 \in [\beta_1;+\infty): \beta_1 \left\langle \mathrm{A}\bar{v}_2, \bar{v}_2 \right\rangle \le \left\langle \mathrm{A}_{\mathrm{II}}\bar{v}_2, \bar{v}_2 \right\rangle \le \beta_2 \left\langle \mathrm{A}\bar{v}_2, \bar{v}_2 \right\rangle \ \forall \bar{v}_2 \in \bar{V}_2.$$

When $\omega = 1$, the continued problem in matrix form is

$$B\bar{u} = \bar{f},\ \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} & 0 \\ 0 & \mathrm{A}_{02} & \mathrm{A}_{23} \\ 0 & \mathrm{A}_{32} & \mathrm{A}_{33} \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{0} \\ \bar{0} \end{bmatrix} = \begin{bmatrix} \bar{f}_1 \\ \bar{0} \\ \bar{0} \end{bmatrix},$$

and the original and fictitious problems in matrix form are

$$\mathrm{A}_{11}\bar{u}_1 = \bar{f}_1,\ \begin{bmatrix} \mathrm{A}_{02} & \mathrm{A}_{23} \\ \mathrm{A}_{32} & \mathrm{A}_{33} \end{bmatrix} \begin{bmatrix} \bar{u}_2 \\ \bar{u}_3 \end{bmatrix} = \begin{bmatrix} \bar{0} \\ \bar{0} \end{bmatrix},\ \begin{bmatrix} \bar{u}_2 \\ \bar{u}_3 \end{bmatrix} = \begin{bmatrix} \bar{0} \\ \bar{0} \end{bmatrix}.$$

When $\omega = \mathrm{II}$, the continued problem in matrix form is

$$B\bar{u} = \bar{f},\ \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} & 0 \\ 0 & \mathrm{A}_{02} & \mathrm{A}_{23} \\ 0 & \mathrm{A}_{32} & \mathrm{A}_{33} \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \end{bmatrix} = \begin{bmatrix} \bar{0} \\ \bar{f}_2 \\ \bar{f}_3 \end{bmatrix},$$

and the original and fictitious problems in matrix form are

$$\begin{bmatrix} \mathrm{A}_{02} & \mathrm{A}_{23} \\ \mathrm{A}_{32} & \mathrm{A}_{33} \end{bmatrix} \begin{bmatrix} \bar{u}_2 \\ \bar{u}_3 \end{bmatrix} = \begin{bmatrix} \bar{f}_2 \\ \bar{f}_3 \end{bmatrix},\ \mathrm{A}_{11}\bar{u}_1 + \mathrm{A}_{12}\bar{u}_2 = \bar{0}.$$

4. Asymptotically optimal analysis of continued system

Let us present a method for analyzing problem (4), if we define the extended matrix as

$$C = \mathrm{A}_{\mathrm{I}} + \gamma \mathrm{A}_{\mathrm{II}},\ \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & C_{23} \\ 0 & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} & 0 \\ \mathrm{A}_{21} & \mathrm{A}_{20} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathrm{A}_{02} & \mathrm{A}_{23} \\ 0 & \mathrm{A}_{32} & \mathrm{A}_{33} \end{bmatrix},\ \gamma \in (0;+\infty),$$

and we assume that

$$\exists\gamma_1 \in (0;+\infty),\ \gamma_2 \in [\gamma_1;+\infty): \gamma_1^2 \left\langle C\bar{v}_2, C\bar{v}_2 \right\rangle \le \left\langle \mathrm{A}_{\mathrm{II}}\bar{v}_2, \mathrm{A}_{\mathrm{II}}\bar{v}_2 \right\rangle \le \gamma_2^2 \left\langle C\bar{v}_2, C\bar{v}_2 \right\rangle \ \forall \bar{v}_2 \in \bar{V}_2, \bar{V}_{\mathrm{II}}$$

$$\exists\alpha \in (0;+\infty): \left\langle \mathrm{A}_{\mathrm{I}}\bar{v}_2, \mathrm{A}_{\mathrm{I}}\bar{v}_2 \right\rangle \le \alpha^2 \left\langle \mathrm{A}_{\mathrm{II}}\bar{v}_2, \mathrm{A}_{\mathrm{II}}\bar{v}_2 \right\rangle \ \forall \bar{v}_2 \in \bar{V}_2, \bar{V}_{\mathrm{II}}.$$

Together with the least residuals approach and the inclusion of additional parameter in the formulation of the extended matrix, we propose method for solving problem (4)

$$\bar{u}^k \in R^N:\ C(\bar{u}^k - \bar{u}^{k-1}) = -\tau_{k-1}(B\bar{u}^{k-1} - \bar{f}),\ k \in \mathrm{N}, \tag{5}$$

$$\tau_{k-1} = \left\langle \bar{r}^{k-1}, \bar{\eta}^{k-1} \right\rangle \Big/ \left\langle \bar{\eta}^{k-1}, \bar{\eta}^{k-1} \right\rangle,\ k \in \mathrm{N}, \forall \bar{u}^0 \in \bar{V}_\omega,\ \alpha < \gamma,\ \text{ if } \omega = 1,\ \text{then } \tau_0 = 1,$$

where the residual vectors, correction vectors and equivalent residual vectors are calculated respectively

$$\bar{r}^{k-1} = B\bar{u}^{k-1} - \bar{f},\ \bar{w}^{k-1} = C^{-1}\bar{r}^{k-1}, k \in \mathrm{N},$$

$$\bar{\eta}^{k-1} = B\bar{w}^{k-1},\ \omega, k \ne 1,\ k \in \mathrm{N}.$$

Let us introduce a norm stronger than the energy norm

$$\|\bar{v}\|_{C^2} = \sqrt{\left\langle C^2\bar{v}, \bar{v} \right\rangle}\ \ \forall \bar{v} \in R^N.$$

Lemma 1. In method (5), if $\omega = 1$, then

$$\left\|\bar{u}^1 - \bar{u}\right\|_{C^2} \le 2\left\|\bar{u}^0 - \bar{u}\right\|_{C^2}.$$

Proof. If error in (5)

$$\bar{\psi}^k = \bar{u}^k - \bar{u},\ k \in \mathrm{N} \cup \{0\},$$

then

$$\left\langle C(\bar{\psi}^1 - \bar{\psi}^0), C(\bar{\psi}^1 - \bar{\psi}^0) \right\rangle = \left\langle -\mathrm{A}_{11}\bar{\psi}_1^0, -\mathrm{A}_{11}\bar{\psi}_1^0 \right\rangle,$$

$$\left\langle C\overline{\psi}^{1}, C\overline{\psi}^{1}\right\rangle - 2\left\langle C\overline{\psi}^{1}, C\overline{\psi}^{0}\right\rangle + \left\langle C\overline{\psi}^{0}, C\overline{\psi}^{0}\right\rangle = \left\langle \mathrm{A}_{11}\overline{\psi}_{1}^{0}, \mathrm{A}_{11}\overline{\psi}_{1}^{0}\right\rangle.$$

We have

$$\left\langle C\overline{\psi}^{0}, C\overline{\psi}^{0}\right\rangle \geq \left\langle \mathrm{A}_{11}\overline{\psi}_{1}^{0}, \mathrm{A}_{11}\overline{\psi}_{1}^{0}\right\rangle,$$

then

$$\left\langle C\overline{\psi},^{1} C\overline{\psi}^{1}\right\rangle - 2\left\langle C\overline{\psi}^{1}, C\overline{\psi}^{0}\right\rangle \leq 0, \left\langle C\overline{\psi}^{1}, C\overline{\psi}^{1}\right\rangle^{2} \leq 4\left\langle C\overline{\psi}^{1}, C\overline{\psi}^{0}\right\rangle^{2} \leq 4\left\langle C\overline{\psi}^{1}, C\overline{\psi}^{1}\right\rangle\left\langle C\overline{\psi}^{0}, C\overline{\psi}^{0}\right\rangle.$$

When reducing

$$\left\langle C\overline{\psi}^{1}, C\overline{\psi}^{1}\right\rangle \leq 4\left\langle C\overline{\psi}^{0}, C\overline{\psi}^{0}\right\rangle, \left\|\overline{\psi}^{1}\right\|_{C^2} \leq 2\left\|\overline{\psi}^{0}\right\|_{C^2}, \left\|\overline{u}^{1} - \overline{u}\right\|_{C^2} \leq 2\left\|\overline{u}^{0} - \overline{u}\right\|_{C^2}.$$

Theorem 1. In method (5) estimates are

$$\omega = 1, \left\|\overline{u}^{k} - \overline{u}\right\|_{C^2} \leq \varepsilon\left\|\overline{u}^{0} - \overline{u}\right\|_{C^2}, \varepsilon = 2(\gamma_2/\gamma_1)(\alpha/\gamma)^{k-1}, k \in \mathrm{N},$$

$$\omega = \mathrm{II}, \left\|\overline{u}^{k} - \overline{u}\right\|_{C^2} \leq \varepsilon\left\|\overline{u}^{0} - \overline{u}\right\|_{C^2}, \varepsilon = (\gamma_2/\gamma_1)(\alpha/\gamma)^{k}, k \in \mathrm{N}.$$

Relative errors in a norm stronger than the energy norm are estimated from above by members of infinitely decreasing geometric progressions.

Proof. Except for $\omega, k = 1,$ we have

$$\overline{\psi}^{k} = \overline{\psi}^{k-1} - \tau_k C^{-1}\mathrm{A}_{\mathrm{II}}\overline{\psi}^{k-1}, \overline{r}^{k} = \overline{r}^{k-1} - \tau_k \mathrm{A}_{\mathrm{II}} C^{-1}\overline{r}^{k-1}.$$

We define

$$\tau_{k-1} = \frac{\left\langle \mathrm{A}_{\mathrm{II}}C^{-1}\overline{r}^{k-1}, \overline{r}^{k-1}\right\rangle}{\left\langle \mathrm{A}_{\mathrm{II}}C^{-1}\overline{r}^{k-1}, \mathrm{A}_{\mathrm{II}}C^{-1}\overline{r}^{k-1}\right\rangle} = \frac{\left\langle \overline{r}^{k-1}, \overline{\eta}^{k-1}\right\rangle}{\left\langle \overline{\eta}^{k-1}, \overline{\eta}^{k-1}\right\rangle}.$$

We have

$$\tau_{k-1} = \frac{\left\langle \mathrm{A}_{\mathrm{II}}C^{-1}\overline{r}^{k-1}, \overline{r}^{k-1}\right\rangle}{\left\langle \mathrm{A}_{\mathrm{II}}C^{-1}\overline{r}^{k-1}, \mathrm{A}_{\mathrm{II}}C^{-1}\overline{r}^{k-1}\right\rangle} = \frac{\left\langle \mathrm{A}_{\mathrm{II}}\overline{w}^{k-1}, C\overline{w}^{k-1}\right\rangle}{\left\langle \mathrm{A}_{\mathrm{II}}\overline{w}^{k-1}, \mathrm{A}_{\mathrm{II}}\overline{w}^{k-1}\right\rangle}.$$

We mark

$$\mathrm{A}_{\mathrm{I}}\overline{w}^{k-1} = \overline{a}, \ \mathrm{A}_{\mathrm{II}}\overline{w}^{k-1} = \overline{b}.$$

We note

$$\tau_k = \frac{\left\langle \overline{b}, \overline{a} + \gamma\overline{b}\right\rangle}{\left\langle \overline{b}, \overline{b}\right\rangle} = \gamma - \frac{\left\langle \overline{a}, \overline{b}\right\rangle}{\left\langle \overline{b}, \overline{b}\right\rangle} \geq \gamma - \frac{\left\langle \overline{a}, \overline{a}\right\rangle^{1/2}\left\langle \overline{b}, \overline{b}\right\rangle^{1/2}}{\left\langle \overline{b}, \overline{b}\right\rangle} \geq \gamma - \frac{\left\langle \overline{a}, \overline{a}\right\rangle^{1/2}}{\left\langle \overline{b}, \overline{b}\right\rangle^{1/2}} \geq \gamma - \alpha > 0.$$

With selected τ_{k-1}

$$\left\langle \overline{r}^{k}, \overline{r}^{k}\right\rangle = \left\langle \overline{r}^{k-1}, \overline{r}^{k-1}\right\rangle - \frac{\left\langle \mathrm{A}_{\mathrm{II}}C^{-1}\overline{r}^{k-1}, \overline{r}^{k-1}\right\rangle^{2}}{\left\langle \mathrm{A}_{\mathrm{II}}C^{-1}\overline{r}^{k-1}, \mathrm{A}_{\mathrm{II}}C^{-1}\overline{r}^{k-1}\right\rangle}.$$

We introduce the relation

$$q_k^2 = \frac{\left\langle \overline{r}^{k}, \overline{r}^{k}\right\rangle}{\left\langle \overline{r}^{k-1}, \overline{r}^{k-1}\right\rangle} = 1 - \frac{\left\langle \mathrm{A}_{\mathrm{II}}C^{-1}\overline{r}^{k-1}, \overline{r}^{k-1}\right\rangle^{2}}{\left\langle \mathrm{A}_{\mathrm{II}}C^{-1}\overline{r}^{k-1}, \mathrm{A}_{\mathrm{II}}C^{-1}\overline{r}^{k-1}\right\rangle\left\langle \overline{r}^{k-1}, \overline{r}^{k-1}\right\rangle} =$$

$$= \frac{\left\langle \mathrm{A}_{\mathrm{II}}\overline{w}^{k-1}, \mathrm{A}_{\mathrm{II}}\overline{w}^{k-1}\right\rangle\left\langle C\overline{w}^{k-1}, C\overline{w}^{k-1}\right\rangle - \left\langle \mathrm{A}_{\mathrm{II}}\overline{w}^{k-1}, C\overline{w}^{k-1}\right\rangle^{2}}{\left\langle \mathrm{A}_{\mathrm{II}}\overline{w}^{k-1}, \mathrm{A}_{\mathrm{II}}\overline{w}^{k-1}\right\rangle\left\langle C\overline{w}^{k-1}, C\overline{w}^{k-1}\right\rangle} = \frac{\left\langle \overline{b}, \overline{b}\right\rangle\left\langle \overline{a} + \gamma\overline{b}, \overline{a} + \gamma\overline{b}\right\rangle - \left\langle \overline{b}, \overline{a} + \gamma\overline{b}\right\rangle^{2}}{\left\langle \overline{b}, \overline{b}\right\rangle\left\langle \overline{a} + \gamma\overline{b}, \overline{a} + \gamma\overline{b}\right\rangle}.$$

We define

$$\left\langle \overline{a}, \overline{a}\right\rangle = a, \left\langle \overline{b}, \overline{b}\right\rangle = b, \left\langle \overline{a}, \overline{b}\right\rangle = z.$$

We have

$$q_k^2 = \frac{ab - z^2}{b(a + \gamma^2 b + 2\gamma z)} \le \max_{|z| \le \sqrt{ab}} q_k^2(z) = q_k^2\left(\frac{-a}{\gamma}\right) = \frac{a}{\gamma^2 b} \le \frac{\alpha^2}{\gamma^2} = q^2,$$

because of

$$q_k^2 \ge 0, \left(q_k^2(z)\right)_z' = \frac{-2\gamma(z + a/\gamma)(z + \gamma b)}{b(a + \gamma^2 b + 2\gamma z)^2}, -\gamma b < -\frac{a + \gamma^2 b}{2\gamma} < -\sqrt{ab} < -\frac{a}{\gamma} < \sqrt{ab}.$$

We obtain

$$\left\langle \bar{r}^k, \bar{r}^k \right\rangle \le q^2 \left\langle \bar{r}^{k-1}, \bar{r}^{k-1} \right\rangle, \left\langle \mathrm{A}_{\mathrm{II}}\bar{\psi}^k, \mathrm{A}_{\mathrm{II}}\bar{\psi}^k \right\rangle \le q^2 \left\langle \mathrm{A}_{\mathrm{II}}\bar{\psi}^{k-1}, \mathrm{A}_{\mathrm{II}}\bar{\psi}^{k-1} \right\rangle.$$

Considering

$$\omega = 1, \left\langle C\bar{\psi}^k,\ C\bar{\psi}^k \right\rangle \le \gamma_1^{-2} \left\langle \mathrm{A}_{\mathrm{II}}\bar{\psi}^k, \mathrm{A}_{\mathrm{II}}\bar{\psi}^k \right\rangle \le \gamma_1^{-2} q^{2(k-1)} \left\langle \mathrm{A}_{\mathrm{II}}\bar{\psi}^1, \mathrm{A}_{\mathrm{II}}\bar{\psi}^1 \right\rangle \le$$

$$\gamma_1^{-2}\gamma_2^2 q^{2(k-1)} \left\langle C\bar{\psi}^1,\ C\bar{\psi}^1 \right\rangle \le 4\gamma_1^{-2}\gamma_2^2 q^{2(k-1)} \left\langle C\bar{\psi}^0,\ C\bar{\psi}^0 \right\rangle,$$

$$\omega = \mathrm{II}, \left\langle C\bar{\psi}^k,\ C\bar{\psi}^k \right\rangle \le \gamma_1^{-2} \left\langle \mathrm{A}_{\mathrm{II}}\bar{\psi}^k, \mathrm{A}_{\mathrm{II}}\bar{\psi}^k \right\rangle \le \gamma_1^{-2} q^{2k} \left\langle \mathrm{A}_{\mathrm{II}}\bar{\psi}^0, \mathrm{A}_{\mathrm{II}}\bar{\psi}^0 \right\rangle \le$$

$$\le \gamma_1^{-2}\gamma_2^2 q^{2k} \left\langle C\bar{\psi}^0,\ C\bar{\psi}^0 \right\rangle,$$

we conclude

$$\omega = 1, \left\langle C\bar{\psi}^k,\ C\bar{\psi}^k \right\rangle \le 4\gamma_1^{-2}\gamma_2^2 q^{2(k-1)} \left\langle C\bar{\psi}^0,\ C\bar{\psi}^0 \right\rangle,$$

$$\omega = \mathrm{II}, \left\langle C\bar{\psi}^k,\ C\bar{\psi}^k \right\rangle \le \gamma_1^{-2}\gamma_2^2 q^{2k} \left\langle C\bar{\psi}^0,\ C\bar{\psi}^0 \right\rangle.$$

Remark 1. If in (5) $\bar{u}^{k-1} = \bar{u}$, then $\bar{u}^k = \bar{u}$.

Proof. In iterative process we obtain

$$\bar{u}^k \in R^N :\ C(\bar{u}^k - \bar{u}) = -\tau_{k-1}(B\bar{u} - \bar{f}),\ k \in \mathrm{N},$$

and

$$\bar{u}^k \in R^N :\ C(\bar{u}^k - \bar{u}) = \bar{0},\ \bar{u}^k - \bar{u} = \bar{0},\ \bar{u}^k = \bar{u},\ k \in \mathrm{N}.$$

5. Algorithmic implementation of method

If $\omega \in W, W = \{1, \mathrm{II}\}$, then algorithm:

1. Value of the squared norm of the initial absolute error

$$E_0 = \left\langle \bar{f}, \bar{f} \right\rangle h^2.$$

2. Initial approximation

$$\forall \bar{u}^0 \in \bar{V}_\omega.$$

3. Residuals

$$\bar{r}^{k-1} = B\bar{u}^{k-1} - \bar{f},\ k \in \mathrm{N}.$$

4. Value of squared norm of absolute error

$$\mathrm{E}_{k-1} = \left\langle \bar{r}^{k-1}, \bar{r}^{k-1} \right\rangle,\ k \in \mathrm{N}.$$

5. Corrections

$$\bar{w}^{k-1} : C\bar{w}^{k-1} = \bar{r}^{k-1},\ k \in \mathrm{N}.$$

6. Equivalent residuals

$$\bar{\eta}^{k-1} = B\bar{w}^{k-1},\ \omega, k \ne 1,\ k \in \mathrm{N}.$$

7. Iterative parameter

$$\tau_{k-1} = \begin{cases} 1, & \omega, k = 1, \\ \left\langle \bar{r}^{k-1}, \bar{\eta}^{k-1} \right\rangle \big/ \left\langle \bar{\eta}^{k-1}, \bar{\eta}^{k-1} \right\rangle, & \omega, k \ne 1,\ k \in \mathrm{N}. \end{cases}$$

8. Next approximation

$$\bar{u}^k = \bar{u}^{k-1} - \tau_{k-1}\bar{w}^{k-1},\ k \in \mathrm{N}.$$

9. Stop criterion based on a given relative error $\delta \in (0;1)$

$$\mathrm{E}_{k-1} \le \mathrm{E}_0 \delta^2,\ k \in \mathrm{N}.$$

If the criterion did not reach, then repeat everything from step 2.

Conclusion

The main results of this work include the development of a method for analysis of a screened harmonic system under Dirichlet and Neumann boundary conditions in a domain with complex geometry, with optimal asymptotics in the number of arithmetic operations. Additionally, the paper presents an asymptotically optimal analysis of the discrete continued screened harmonic system and introduces an algorithm that implements the method for analyzing screened harmonic systems with optimal asymptotics in the number of arithmetic operations. The work contributes to the field of analysis of screened harmonic systems and provides a method for solving boundary value problems for the screened Poisson equation in areas with complex geometry.

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АНАЛИЗ ЭКРАНИРОВАННОЙ ГАРМОНИЧЕСКОЙ СИСТЕМЫ ПРИ КРАЕВЫХ УСЛОВИЯХ ДИРИХЛЕ И НЕЙМАНА

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Аннотация. Рассматривается экранированная гармоническая система с граничными условиями Дирихле и Неймана в области со сложной геометрией и предлагается метод анализа такой системы. Разработка этого метода особенно актуальна для решения краевых задач для экранированного уравнения Пуассона в областях со сложной геометрией, которые используются для описания различных физических систем в механике, гидродинамике, электротехнике и теплотехнике. Предложенный алгоритм анализа экранированной гармонической системы при этих граничных условиях вносит существенный вклад в эту область. Предложенный метод включает продолжение экранированной гармонической системы через границы с условиями Дирихле и Неймана. Затем продолжение дискретизируется системой линейных алгебраических уравнений. Проводится асимптотически оптимальный анализ дискретной продолженной экранированной гармонической системы и алгоритм, реализующий метод анализа экранированной гармонической системы с оптимальной асимптотикой по числу арифметических операций.

Ключевые слова: экранированная гармоническая система; асимптотически оптимальный анализ.

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