DOI: 10.14529/mmph250302

GENERALIZED SOLVABILITY OF INITIAL-BOUNDARY VALUE PROBLEMS FOR QUASIHYDRODYNAMIC SYSTEM OF EQUATIONS IN WEIGHTED SOBOLEV SPACES

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Abstract. The paper considers the analog of the first initial boundary value problem for a quasihydrodynamic system of equations in the case of a weakly compressible fluid in weighted Sobolev spaces. The system is an elliptic-parabolic system: its first part is an elliptic equation for the pressure gradient, and its second part is a parabolic system for the velocity vector. The unknown variables of the pressure gradient and velocity vector belong to the principal parts of the elliptic equation and the parabolic system. The fixed part of the system is not uniformly elliptic, thus complicating the study of the problem. T.G. Elizarova and B.N. Chetverushkin introduce the system by averaging the known kinetic model. The first versions of the system are the system of quasi-gasodynamic equations. Later, Y.V. Sheretov, based on a more general equation of state, obtains another model, which is called quasihydrodynamic system of equations, and thoroughly analyses its properties. However, the issues of generalized solvability of initial boundary value problems for such systems have not been studied in detail yet. There are only some partial results. The paper aims to fill this gap. We prove generalized solvability of the system in some weight classes characterizing the behavior of solutions at *t*→∞ according to the Galerkin method and the obtained prior estimates. The decreasing (growing) behavior of the solution depends on the decreasing (growing) right-hand side of the system. The decrease (growth) at *t*→∞ of the used weight functions can be both exponential and power.

Keywords: Initial-boundary value problem; quasihydrodynamic system; prior estimates; weight functions; existence theorem.

Introduction. We consider a quasihydrodynamic system of equations in the case of a weakly compressible fluid:

$$\begin{aligned} &\operatorname{div}\vec{u} = \operatorname{div}\vec{w},\ \vec{w} = \tau\left((\vec{u},\nabla)\vec{u} + \frac{1}{\rho}\nabla p - \vec{f} \right), (t,x) \in Q = (0,\infty)\times G, G \subset \mathbb{R}^3, \\ &\frac{\partial \vec{u}}{\partial t} + (\vec{u}-\vec{w},\nabla)\vec{u} + \frac{1}{\rho}\nabla p = \vec{f} + \mu\Delta\vec{u} + \mu\nabla(\vec{u}) + (\vec{u},\nabla)\vec{w} + \vec{w}\operatorname{div}\vec{u}, \mu = \eta/\rho. \end{aligned} \tag{1}$$

where G is bounded domain with boundary $\Gamma \in C^2$, $\mu = \eta/\rho$ is kinematic viscosity coefficient. Density ρ, dynamic viscosity μ and characteristic relaxation time τ are positive constants. Vector field $\vec{f} = \vec{f}(x,t)$ determines the mass density of external forces. The system (1) is closed with respect to the unknown functions, i. e. the velocity vector $\vec{u} = \vec{u}(x,t)$ and the pressure $p = p(x,t)$. Symbols div and ∇ denote the divergence and gradient, respectively.

We look for a solution to the system (1) satisfying the following initial and boundary conditions and the normalization condition:

$$\vec{u}\big|_{\Gamma} = 0, \vec{w}\cdot\nu\big|_{\Gamma} = 0, \vec{u}_{t=0} = \vec{u}_0(x), \int_G p(t,x)\,dx = 0, \tag{2}$$

where ν is the unit vector of the outward normal to Γ.

The system (1) in a more general form was derived in [1, 2] by averaging the known kinetic model. The first variants of the system are called the system of quasigasodynamic equations. The derivation of the system and some results can be found in the monographs [3, 4]. Later, using a more general equation of state, another model known as quasihydrodynamic system of equations was developed in [5, 6]. In particular, a detailed analysis of the properties of this model is presented in the monograph [7]. Here, for a quasihydrodynamic system of equations in the case of a weakly compressible fluid (i.e., for system (1)), dissipative properties are investigated and a theorem of uniqueness of the classical solution to the main initial-boundary value problem is proven. Zlotnik A.A. in [8] constructed a system with a general regularizing velocity on the basis of a linearized (on a constant solution) quasihydrodynamic system of equations and established the degeneration of the parabolicity property of the original system. Later, for the first time, he constructed a quasigasodynamic type regularization of the heterogeneous model (in the quasihomogeneous form), for which the difference scheme an explicit two-layer in time and symmetric three-point in space in the 1D case was constructed [9]. In [10], a model based on quasigasodynamic and quasihydrodynamic equations in multiscale media is investigated, which can be used in applications to porous media theory. A computational multiscale method based on the idea of bond energy minimization was proposed to solve quasigasodynamics problems and improve the accuracy of simulations. Recently, regularized hydrodynamic equations of quasihydrodynamic type have been used in the numerical solution of a number of practical problems. Relevant results are exposed in [11–14]. Note that the system (1) is an elliptic-parabolic system and both equations, elliptic and parabolic, contain the senior derivatives of the unknown pressures p and the velocity vector $\vec{u}$. Such systems often arise in applications. In [15], to solve the problem of two-phase non-isothermal filtration, the authors consider a system consisting of one elliptic and two parabolic equations with known boundary conditions. The authors in [16], using the technique of Fourier multipliers, proved an a priori estimate for strong solutions to elliptic-parabolic equations of mixed type in Sobolev space. In [17], a family of models for the flow and transport of multiscale single-phase fluid in inhomogeneous porous media based on an elliptic-parabolic system consisting of an elliptic equation for steady-state flow and a parabolic equation for transient advection-diffusion is described. The existence and uniqueness theorems of generalized and regular solutions of an analog of the first initial boundary value problem for the system (1) are presented in [18, 19], respectively. The proof of the existence of generalized solution in [18] is based on the Galerkin method and a priori estimates. In [19], under certain conditions on the data, it is shown that there exists a unique regular solution of the initial-boundary value problem locally in time. The existence and uniqueness theorems for generalized and regular solutions to an initial-boundary value problems for a quasihydrodynamic system in the linearized case are presented in [20]. In the case of a regular solution, there are some restrictions on the norms of the data. The obtained results provide appropriate stability estimates for solutions to the original nonlinear problem.

In this paper we study the solvability of initial-boundary value problems for the system (1) in some weight classes characterizing the behavior of generalized solutions as $t \to \infty$.

Preliminaries

Let $\vec{u}, p$ be a sufficiently smooth solution to the problem (1), (2). We say that $\vec{u} \in L_{2,loc}(0,\infty;E)$ (E is a Banach space), if $\vec{u} \in L_2(0,T;E)$ for any $T < \infty$. Let $\psi(t)$ be a non-negative function such that $\mu(\{t : \psi(t) = 0\}) = 0$. Here μ is the Lebesgue measure. By $L_{2,\psi}(0,\infty;E)$, we mean the space of measurable functions $\vec{u}(t)$ such that $\psi\vec{u} \in L_2(0,\infty;E)$. Let us multiply the first and second equation of the system by the functions φ and $\vec{\psi}$ respectively such that $\varphi \in L_2\left(0,\infty;W_2^1(G)\right)$, $\int\limits_G \varphi(x)dx = 0$, $\vec{\psi} \in L_2\left(0,\infty;W_2^1(G)\right)$, $\vec{\psi}\big|_S = 0$, φ and $\vec{\psi}$ have bounded supports. Integrating the results over G, we arrive at the equalities:

$$\int\limits_G \vec{u}\cdot\nabla\varphi dx = \int\limits_G \vec{w}\cdot\nabla\varphi dx = \tau\left((\vec{u},\nabla)\vec{u},\nabla\varphi\right) + \frac{\tau}{\rho}(\nabla p,\nabla\varphi) - \tau\left(\vec{f},\nabla\varphi\right),$$

$$\left(\frac{\partial\vec{u}}{\partial t},\vec{\psi}\right) + \left(\left(\left(\vec{u}-\vec{w}\right),\nabla\right)\vec{u},\vec{\psi}\right) + \frac{1}{\rho}(\nabla p,\vec{\psi}) = \left(\vec{f},\vec{\psi}\right) - \mu(\nabla\vec{u},\nabla\vec{\psi}) - \tag{3}$$

$$\mu\left(\vec{u},\operatorname{div}\vec{\psi}\right) + \left((\vec{u},\nabla)\vec{w},\vec{\psi}\right) + \left(\vec{w}\operatorname{div}\vec{u},\vec{\psi}\right),$$

where the point $\cdot$ means the scalar product in R^3 and $(u,v)=\int\limits_G uvdx$ for scalar functions and $(\vec{u},\vec{v})=\int\limits_G \vec{u}\cdot\vec{v}dx$ for vector functions. Integrating by parts, we have

$$\left((\vec{u},\nabla)\vec{w},\vec{\psi}\right)=\int\limits_G(\vec{u},\nabla)\vec{w}\cdot\vec{\psi}dG=-\int\limits_G \mathrm{div}\vec{u}\vec{w}\cdot\vec{\psi}dG-\int\limits_G(\vec{u},\nabla)\vec{\psi}\cdot\vec{w}dG. \quad (4)$$

Using this equality in (3), we obtain the equations

$$(\vec{u},\nabla\varphi)=(\vec{w},\nabla\varphi),\left(\frac{\partial\vec{u}}{\partial t},\vec{\psi}\right)-\left((\vec{u}-\vec{w},\nabla)\vec{\psi},\vec{u}\right)+\frac{1}{\rho}(\nabla p,\vec{\psi})+$$
$$\mu(\nabla\vec{u},\nabla\vec{\psi})+\mu(\mathrm{div}\vec{u},\mathrm{div}\vec{\psi})+\left((\vec{u},\nabla)\vec{\psi},\vec{w}\right)=\left(\vec{f},\vec{\psi}\right), \quad (5)$$

valid for almost all t. Equalities (5) can serve as a basis for defining a generalized solution to the problem. Let $p_0\in[1,3/2]$, $q_0=2p_0/(4p_0-3)$, $p_1=5/4$. Then $q_0\in[1,2]$. Functions $\vec{u}\in L_{2,loc}\left(0,\infty;W_2^1(G)\right)\cap L_{\infty,loc}\left(0,\infty;L_2(G)\right)$, $u_t\in L_{p_1,loc}\left(0,\infty;W_{p_1}^{-1}(G)\right)$, $p\in L_{p_1}\left(0,\infty;W_{p_1}^1(G)\right)$ such that $\frac{\nabla p}{\rho}+(\vec{u},\nabla)\vec{u}\in L_{2,loc}\left(0,\infty,L_2(G)\right)$, satisfying (2) are called generalized solutions of the problem (1), (2) if

$$\int\limits_0^\infty(\vec{u},\nabla\varphi)dt=\int\limits_0^\infty(\vec{w},\nabla\varphi)dt,\ \int\limits_0^\infty\Big[(\frac{\partial\vec{u}}{\partial t},\vec{\psi})-((\vec{u}-\vec{w},\nabla)\vec{\psi},\vec{u}+\frac{1}{\rho}(\nabla p,\vec{\psi})+$$
$$\mu(\nabla\vec{u},\nabla\vec{\psi})+\mu(\mathrm{div}\vec{u},\mathrm{div}\vec{\psi})+\left((\vec{u},\nabla)\vec{\psi},\vec{w}\right)dt=\int\limits_0^\infty\left(\vec{f},\vec{\psi}\right)dt,$$

for all functions $\varphi\in L_2\left(0,\infty;W_2^1(G)\right)$ with $\int_G\varphi(t,x)dx=0,\vec{\psi}\in L_5\left(0,\infty;W_5^1(G)\right)$ and $\vec{\psi}\big|_S=0$, having a bounded support in t. Let

$$a\left(\vec{u},\vec{\psi}\right)=\left(\frac{\partial\vec{u}}{\partial t},\vec{\psi}\right)-\left((\vec{u}-\vec{w},\nabla)\vec{\psi},\vec{u}\right)+\frac{1}{\rho}(\nabla p,\vec{\psi})+\mu(\nabla\vec{u},\nabla\vec{\psi})+\ \mu(\vec{u},\mathrm{div}\vec{\psi})+\left((\vec{u},\nabla)\vec{\psi},\vec{w}\right).$$

The main results

Let's introduce an auxiliary weight function. We consider several different cases. In the former case $\beta(t)=e^{\gamma t}$ $(\gamma\neq 0,\gamma<\mu/2\delta_0)$, where δ_0 is the constant δ_0 from Poincaré's inequality

$$\int\limits_G|\vec{u}|^2dx\le\delta_0\int\limits_G|\nabla\vec{u}|^2dx,$$

valid for all $\vec{u}\in W_2^1(G)$, such that $\vec{u}\big|_\Gamma=0$. In the second case $\beta(t)=(M+t)^\gamma$ $(\gamma\neq 0)$, where $M>0$ is some constant, which will be chosen below.

Theorem. Let $f\sqrt{\beta}\in L_2(Q)$, $u_0\in L_2(G)$. Then there exists a generalized solution of the problem (1), (2) such that $\sqrt{\beta}u\in L_2\left(0,\infty;W_2^1(G)\right)$, $\sqrt{\beta}u\in L_\infty\left(0,\infty;L_2(G)\right)$, $\beta(u,\nabla)u\in L_2(Q)$, $\sqrt{\beta}\left(\frac{\nabla p}{\rho}+(\vec{u},\nabla)\vec{u}\right)\in L_2(Q)$, $\beta(u,\nabla)u\in L_{q_0}\left(0,\infty;L_{p_0}(G)\right)$ for any $p_0\in[1,3/2]$, $\nabla p\beta^\alpha\in L_{q_0}\left(0,\infty;L_{p_0}(G)\right)$, $\vec{u}_t\beta^\alpha\in L_{q_0}\left(0,\infty;W_{p_0}^{-1}(G)\right)$ for any $p_0\in[1,3/2]$ and for all α such that $\alpha<1/2$ if $\beta=e^{\gamma t}$ $(\gamma>0)$ and $\alpha<\frac{1}{2}-\frac{2-q_0}{2|\beta|q_0}$ if $\beta=(t+M)^{-\gamma},\beta<0$; the case of $\alpha=\frac{1}{2}$ is possible when $q_0=2$; $\alpha>\frac{1}{2}+\frac{2-q_0}{2|\beta|q_0}$ and $\alpha\ge 1$ if $\beta=(t+M)^{-\gamma},\beta>0$; $\alpha\ge 1$ if $\beta=e^{\gamma t}$ $(\gamma<0)$.

Proof. First, we derive the first a priori estimate with weight β for smooth solutions to the problem. Take $\varphi = p\beta$, $\vec{\psi} = \beta\vec{u}$ in the definition of a generalized solution. We obtain the equalities

$$\tau\left((\vec{u},\nabla)\vec{u},\beta\nabla p\right)+\frac{\tau}{\rho}(\nabla p,\beta\nabla p)-\tau\left(\vec{f},\beta\nabla p\right)-(\vec{u},\beta\nabla p)=0,$$

$$\left(\frac{\partial\vec{u}}{\partial t},\vec{u}\beta\right)-\left(\left(\vec{u}-\vec{w},\nabla\right)\vec{u},\vec{u}\beta\right)+\frac{1}{\rho}(\nabla p,\vec{u}\beta)+\mu\left(\nabla\vec{u},\nabla\vec{u}\beta\right)+\mu\left(\operatorname{div}\vec{u},\operatorname{div}\vec{u}\beta\right)+ \tag{6}$$

$$\left((\vec{u},\nabla)\vec{u}\beta,\left(\tau(\vec{u},\nabla)\vec{u}+\frac{\tau}{\rho}\nabla p-\tau\vec{f}\right)\right)=\left(\vec{f},\vec{u}\beta\right).$$

Dividing the first equality in (6) by ρ and adding it to the second equality and using the equality $\left(\left(\vec{u}-\vec{w},\nabla\right)\vec{u},\vec{u}\right)=0$, we obtain that

$$\frac{\partial}{\partial t}\int_G\frac{|\vec{u}|^2\beta}{2}dx-\int_G\frac{|\vec{u}|^2\beta_t}{2}dx+\mu\left(\nabla\vec{u},\nabla\vec{u}\beta\right)+\mu\left(\operatorname{div}\vec{u},\operatorname{div}\vec{u}\beta\right)+\frac{\tau}{\rho^2}(\nabla p,\beta\nabla p)+$$

$$\frac{\tau}{\rho}\left((\vec{u},\nabla)\vec{u},\beta\nabla p\right)-\frac{\tau}{\rho}\left(\vec{f},\beta\nabla p\right)-\frac{1}{\rho}(\vec{u},\beta\nabla p)+\frac{1}{\rho}\left(\nabla p,\vec{u}\beta\right)+\tau\left((\vec{u},\nabla)\vec{u},(\vec{u},\nabla)\vec{u}\beta\right)+ \tag{7}$$

$$\frac{\tau}{\rho}\left(\nabla p,(\vec{u},\nabla)\vec{u}\beta\right)-\tau\left((\vec{u},\nabla)\vec{u},\vec{f}\beta\right)=\left(\vec{f},\vec{u}\beta\right).$$

Transforming this equality, we conclude that

$$\frac{\partial}{\partial t}\int_G\frac{(|\vec{u}|^2\beta)}{2}dx-\int_G\frac{(|\vec{u}|^2\beta_t)}{2}dx+\mu\left(\nabla\vec{u},\nabla\vec{u}\beta\right)+\mu\left(\operatorname{div}\vec{u},\operatorname{div}\vec{u}\beta\right)+$$

$$\frac{\tau}{\rho^2}(\nabla p,\nabla p,\beta)+\tau\left((\vec{u},\nabla)\vec{u},(\vec{u},\nabla)\vec{u}\beta\right)+\frac{2\tau}{\rho}\left((\vec{u},\nabla)\vec{u},\nabla p,\beta\right)- \tag{8}$$

$$\frac{\tau}{\rho}\left(\vec{f},\nabla p,\beta\right)-\tau\left((\vec{u},\nabla)\vec{u},\vec{f},\beta\right)=\left(\vec{f},\vec{u}\beta\right),$$

$$\frac{\partial}{\partial t}\int_G\frac{|\vec{u}|^2\beta}{2}dx-\int_G\frac{|\vec{u}|^2\beta_t}{2}dx+\mu\left(\nabla\vec{u},\nabla\vec{u}\beta\right)+\mu\left(\operatorname{div}\vec{u},\operatorname{div}\vec{u}\beta\right)+$$

$$\tau\left(\beta\left(\frac{\nabla p}{\rho}+(\vec{u},\nabla)\vec{u}\right),\left(\frac{\nabla p}{\rho}+(\vec{u},\nabla)\vec{u}\right)\right)=\tau\left(\vec{f},\frac{\nabla p}{\rho}\beta+(\vec{u},\nabla)\vec{u}\beta\right)+\left(\vec{f},\vec{u}\beta\right). \tag{9}$$

Estimate the right-hand side using the Cauchy inequality

$$|ab|\le\frac{\varepsilon}{2}a^2+\frac{1}{2\varepsilon}b^2,(\varepsilon>0). \tag{10}$$

We have that

$$\tau\left|(\vec{f},\frac{\nabla p}{p}\beta+(\vec{u},\nabla)\vec{u}\beta)\right|\le\tau\int_G\frac{|\vec{f}|^2\beta}{2}dx+\frac{\tau}{2}\int_G\left|\frac{\nabla p}{p}+(\vec{u},\nabla)\vec{u}\right|^2\beta dx \tag{11}$$

$$\left|(\vec{f},\vec{u}\beta)\right|\le\frac{\varepsilon}{2}\int_G|\vec{u}|^2\beta dx+\int_G|\vec{f}|^2\beta dx\frac{1}{2\varepsilon}. \tag{12}$$

Using these inequalities in (9), we obtain

$$\frac{\partial}{\partial t}\int_G\frac{|\vec{u}|^2\beta}{2}dx-\int_G\frac{|\vec{u}|^2\beta_t}{2}dx+\mu\left(\nabla\vec{u},\nabla\vec{u}\beta\right)+\mu\left(\operatorname{div}\vec{u},\operatorname{div}\vec{u}\beta\right)+$$

$$\frac{\tau}{2}\left(\beta\left(\frac{\nabla p}{\rho}+(\vec{u},\nabla)\vec{u}\right),\left(\frac{\nabla p}{\rho}+(\vec{u},\nabla)\vec{u}\right)\right)\le\frac{\tau}{2}\int_G|\vec{f}|^2\beta dx+\frac{1}{2\varepsilon}\int_G|\vec{f}|^2\beta dx+\frac{\varepsilon\delta_0}{2}\int_G|\nabla\vec{u}|^2\beta dx. \tag{13}$$

Let's take $\varepsilon=\mu/\delta_0$. We conclude that

$$\frac{\partial}{\partial t}\int_G \frac{|\vec{u}|^2\beta}{2}dx-\int_G \frac{|\vec{u}|^2\beta_t}{2}dx+\frac{\mu}{2}\left(\nabla\vec{u},\nabla\vec{u}\beta\right)+\mu\left(\operatorname{div}\vec{u},\operatorname{div}\vec{u}\beta\right)+$$
$$\frac{\tau}{2}\left(\beta\left(\frac{\nabla p}{\rho}+\left(\vec{u},\nabla\right)\vec{u}\right),\left(\frac{\nabla p}{\rho}+\left(\vec{u},\nabla\right)\vec{u}\right)\right)\le\int_G|\vec{f}|^2\beta dx\left(\frac{\tau}{2}+\frac{\delta_0}{2\mu}\right). \quad (14)$$

Consider the following two cases: a) $\beta_t<0$, b) $\beta_t>0$. Let $J=\frac{\mu}{4}\left(\nabla\vec{u},\nabla\vec{u}\beta\right)+\mu\left(\operatorname{div}\vec{u},\operatorname{div}\vec{u}\beta\right)+\frac{\tau}{2}\left(\beta\left(\frac{\nabla p}{\rho}+\left(\vec{u},\nabla\right)\vec{u}\right),\left(\frac{\nabla p}{\rho}+\left(\vec{u},\nabla\right)\vec{u}\right)\right)$. Integrating (14) from 0 to t, in case a) we obtain

$$\int_G \frac{|\vec{u}|^2\beta(t)}{2}dx+\int_0^t\int_G J\,dxdt\le C_1\int_0^t\int_G|\vec{f}|^2\beta dxdt+\int_G\frac{u_0^2}{2}\beta(0)dx=M. \quad (15)$$

In the case b), we can rewrite the inequality (14) in the form

$$\frac{\partial}{\partial t}\int_G \frac{|\vec{u}|^2\beta(t)}{2}dx+\int_G|\vec{u}|^2\left(\frac{\mu\beta}{4\delta_0}-\frac{\beta_t}{2}\right)+J dx\le\int_G|\vec{f}|^2\beta dx\left(\frac{\tau}{2}+\frac{\delta_0}{2\mu}\right). \quad (16)$$

Let $\beta(t)=e^{\gamma t}$ $(\gamma>0)$. Since $\gamma\le\mu/2\delta_0$, (16) implies that the inequality (15) holds. Let $\beta=(t+M)^{-\gamma}$ $(\gamma<0)$. In this case, choosing a sufficiently large number of M ($\frac{\mu}{2\delta_0}+\frac{\gamma}{M}>0$), we obtain the inequality $\frac{\mu\beta}{4\delta_0}-\frac{\beta_t}{2}\ge 0$ which validates the inequality (15). The inequality (15) yields

$$\max_t\int_G|\vec{u}|^2\beta dx\le M,\int_G|\vec{u}|^2(t,x)dx\le\frac{M}{\beta(t)}, \quad (17)$$

$$\int_Q\frac{\mu}{4}\left(\nabla\vec{u},\nabla\vec{u}\beta\right)+\mu(\operatorname{div}\vec{u})^2\beta+\frac{\tau}{2}(\beta(\frac{\nabla p}{p}+(\vec{u},\nabla)\vec{u}),(\frac{\nabla p}{p}+(\vec{u},\nabla)\vec{u}))\le M. \quad (18)$$

As a consequence, we obtain the following a priori estimates for solutions:

$$\left\|\vec{u}\sqrt{\beta}\right\|_{L_2\left(0,\infty;W_2^1(G)\right)}+\left\|\operatorname{div}\vec{u}\sqrt{\beta}\right\|_{L_2(Q)}+\left\|\frac{\nabla p}{\rho}\sqrt{\beta}+\left(\vec{u},\nabla\right)\vec{u}\sqrt{\beta}\right\|_{L_2(Q)}\le C_1(M), \quad (19)$$

where $C_1(M)$ is some constant depending on M,μ,τ,

$$\left\|\vec{u}\sqrt{\beta}\right\|_{L_\infty\left(0,\infty;L_2(G)\right)}\le C_1(M). \quad (20)$$

Next, we evaluate all summands included in the definition of a generalized solution. Demonstrate that

$$\left\|\beta\left(\vec{u},\nabla\right)\vec{u}\right\|_{L_{q_0}\left(0,\infty;L_{p_0}(G)\right)}\le C,\, p_0\in[1,3/2], \quad (21)$$

where the constant C has the same properties as the constant C_1. The Hölder inequality yields

$$\left\|\beta\left(\vec{u},\nabla\right)\vec{u}\right\|_{L_{p_0}(G)}\le C\left\|\nabla\vec{u}\beta^{1/2}\right\|_{L_2(G)}\cdot\left\|\vec{u}\beta^{1/2}\right\|_{L_{p_0 q}(G)}, \quad (22)$$

where $q=\frac{2}{2-p_0}$. Next, we use the embedding $W_2^s(G)\subset L_{q_0 p}(G)$ for $p_0 q=r=\frac{6}{3-2s}$, in this case

$$\frac{p_0}{2-p_0}=\frac{3}{3-2s}=s=\frac{3(p_0-1)}{p_0}.$$

The necessary inequality $s\le 1$ is equivalent to the inequality $p_0\le 3/2$. From (22) it follows that

$$\left\|\beta\left(\vec{u},\nabla\right)\vec{u}\right\|_{L_{p_0}(G)}\le C_1\left\|\nabla\vec{u}\beta^{1/2}\right\|_{L_2(G)}\cdot\left\|\vec{u}\beta^{1/2}\right\|_{W_2^s(G)}. \quad (23)$$

We estimate the last multiplier using the interpolation inequality [21]

$$\left\|\vec{u}\sqrt{\beta}\right\|_{W_2^s(G)} \le C\left\|\vec{u}\sqrt{\beta}\right\|_{W_2^1(G)}^{\theta}\left\|\vec{u}\sqrt{\beta}\right\|_{L_2(G)}^{1-\theta}, \tag{24}$$

where $s=\theta$. From (23) we obtain the estimate

$$\left\|\beta(\vec{u},\nabla)\vec{u}\right\|_{L_{p_0}(G)} \le C\left\|\sqrt{\beta}\vec{u}\right\|_{W_2^1(G)}^{1+s}\cdot\left\|\sqrt{\beta}\vec{u}\right\|_{L_2(G)}^{1-s}. \tag{25}$$

Using (20), we obtain

$$\left\|\beta(\vec{u},\nabla)\vec{u}\right\|_{L_{q_0}(0,\infty;L_{p_0}(G))} \le C_1(M)\left(\int_0^\infty\|\beta\vec{u}\|_{W_2^1(G)}^{q_0(1+s)}dt\right)^{1/q_0} \le C_2(M), \tag{26}$$

where we choose

$$q_0(1+s)=2,\ \text{i.e. } q_0=2p_0/(4p_0-3). \tag{27}$$

By definition, $q_0\ge 1$. The estimate (19) yields

$$\left\|\frac{\nabla p}{\rho}\sqrt{\beta}+\sqrt{\beta}(\vec{u},\nabla)\vec{u}\right\|_{L_2(Q)} \le C(M). \tag{28}$$

Let $g=(\dfrac{\nabla p}{\rho}+(\vec{u},\nabla)\vec{u})$. Next, we infer

$$\left\|\beta^{\alpha}\frac{\nabla p}{\rho}\right\|_{L_{q_0}(0,\infty;L_{p_0}(G))} \le \left\|\beta^{\alpha}g\right\|_{L_{q_0}(0,\infty;L_{p_0}(G))}+\left\|\beta^{\alpha}(\vec{u},\nabla)\vec{u}\right\|_{L_{q_0}(0,\infty;L_{p_0}(G))}. \tag{29}$$

Let $\beta_t>0$ and let $\alpha\le 1$. In this case

$$\left\|\beta^{\alpha}(\vec{u},\nabla)\vec{u}\right\|_{L_{q_0}(0,\infty;L_{p_0}(G))} \le \left\|\beta(\vec{u},\nabla)\vec{u}\right\|_{L_{q_0}(0,\infty;L_{p_0}(G))} \le C(M). \tag{30}$$

If $\beta_t<0$ and $\alpha\ge 1$ then similarly we have that

$$\left\|\beta^{\alpha}(\vec{u},\nabla)\vec{u}\right\|_{L_{q_0}(0,\infty;L_{p_0}(G))} \le \left\|\beta(\vec{u},\nabla)\vec{u}\right\|_{L_{q_0}(0,\infty;L_{p_0}(G))} \le C(M). \tag{31}$$

Next, we derive that

$$\left\|\beta^{\alpha}g\right\|_{L_{q_0}(0,\infty;L_{p_0}(G))} \le \left(\int_0^\infty\left(\int_G|g|^{p_0}\,dx\right)^{q_0/p_0}\beta^{\alpha q_0}dt\right)^{1/q_0}, \tag{32}$$

where

$$\left(\int_G|g|^{p_0}\,dx\right)^{q_0/p_0} \le \left(\int_G|g|^2\,dx\right)^{q_0/2}c_0,\ \ c_0=\mu(G)^{q_0(1/p_0-1/2)}, \tag{33}$$

with μ is the Lebesgue measure. In this case, we infer

$$\left\|\beta^{\alpha}g\right\|_{L_{q_0}(0,\infty;L_{p_0}(G))} \le c_1\left(\int_0^\infty\left(\int_G|g|^2\,dx\right)^{\frac{q_0}{2}}\beta^{\frac{q_0}{2}}\beta^{\alpha q_0-\frac{q_0}{2}}dt\right)^{\frac{1}{q_0}} \le c_1\left(\int_0^\infty\beta\int_G|g|^2\,dxdt\right)^{\frac{1}{2}}\left(\int_0^\infty\beta^{\frac{2q_0}{2-q_0}(\alpha-\frac{1}{2})}dt\right)^{\frac{2-q_0}{2q_0}}, \tag{34}$$

Let $\beta=e^{\gamma t},\gamma>0.$ In this case, for convergence of the last integral, it is necessary that $\alpha<1/2$. If $\beta=(t+M)^{-\gamma},\gamma<0$, then it is necessary that

$$\alpha<\frac{1}{2}-\frac{(2-q_0)}{2|\gamma|q_0}. \tag{35}$$

Note that the inequality $\alpha\ge 0$ is satisfied under the condition that $2/(1+|\gamma|)\le q_0\le 2$. Let $\beta=e^{\gamma t},\gamma<0.$ For convergence of the integral, it is necessary that $\alpha>1/2$. If $\beta=(t+M)^{-\gamma},\gamma>0$, it is necessary that

$$\alpha>\frac{(2-q_0)}{2|\gamma|q_0}+\frac{1}{2}. \tag{36}$$

Note that the inequality $\frac{(2-q_0)}{2|\gamma|q_0}+\frac{1}{2}\geq 1$ is fulfilled whenever if $\frac{2}{1+|\gamma|}\geq q_0$. If $\beta_t>0$, then the inequalities (29), (30) imply the estimate

$$\left\|p\beta^{\alpha}\right\|_{L_{q_0}(0,\infty;L_{p_0}(G))}\leq C(M), \tag{37}$$

where $\alpha<1/2$ if $\beta=e^{\gamma t}$ $(\gamma>0)$ and $\alpha<\frac{1}{2}-\frac{2-q_0}{2|\gamma|q_0}$ if $\beta=(t+M)^{-\gamma},\gamma>0$. The case $\alpha=\frac{1}{2}$ is possible when $q_0=2$. If $\beta_t<0$, the inequalities (29), (31) yield

$$\left\|p\beta^{\alpha}\right\|_{L_{q_0}(0,\infty;L_{p_0}(G))}\leq C(M), \tag{38}$$

where $\alpha>\frac{1}{2}+\frac{2-q_0}{2|\gamma|q_0}$ and $\alpha\geq 1$. If $\frac{2}{1+|\gamma|}\geq q_0$ then the inequality $\frac{1}{2}+\frac{2-q_0}{2|\gamma|q_0}\geq 1$ holds. We have obtained the estimate

$$\left\|\beta^{\alpha}\nabla p\right\|_{L_{q_0}(0,\infty;L_{p_0}(G))}+\left\|\beta(\vec{u},\nabla)\vec{u}\right\|_{L_{q_0}(0,\infty;L_{p_0}(G))}\leq C_4(M), \tag{39}$$

As a consequence, if $p_0=p_1=5/4$ ($p_0=q_0$ in this case) then we conclude that

$$\left\|\beta^{\alpha}\nabla p\right\|_{L_{p_1}(Q)}+\left\|\beta(\vec{u},\nabla)\vec{u}\right\|_{L_{p_1}(Q)}\leq C_4(M). \tag{40}$$

Since

$$\vec{w}=\tau\left(\frac{\nabla p}{\rho}+(\vec{u},\nabla)\vec{u}-\vec{f}\right), \tag{41}$$

we have the following inequality for the norm of $\vec{w}$:

$$\left\|\sqrt{\beta}\vec{w}\right\|_{L_2(Q)}\leq\left\|\sqrt{\beta}\left(\frac{\nabla p}{\rho}+(\vec{u},\nabla)\vec{u}\right)\right\|_{L_2(Q)}+\left\|\sqrt{\beta}\vec{f}\right\|_{L_2(Q)}\leq C_5(M). \tag{42}$$

Let us estimate the summands from the definition of a generalized solution. We have that

$$\left((\vec{u}-\vec{w},\nabla)\vec{\psi},\vec{u}\right)=\int_G\sum_{i,j}(u_i-w_i)\psi_{jx_i}u_j dx. \tag{43}$$

Consider the functional $l(\vec{\psi})$. The Hölder inequality yields

$$I=\int_G w_i\psi_{jx_i}u_j dx,\ |I|\leq\left\|w_iu_j\right\|_{L_{p_0}(G)}\cdot\left\|\psi_{jx_i}\right\|_{L_{p_0'}(G)},1/p_0+1/p_0'=1. \tag{44}$$

Further, we obtain (see (20)) that

$$[\int_G|w_i|^{p_0}(u_j)^{p_0}dx]^{\frac{1}{p_0}}\leq\left\|w_i\right\|_{L_2(G)}\cdot\left\|u_j\right\|_{L_{\frac{2p_0}{2-p_0}}(0,\infty,L_{q_0}(G))}\leq$$

$$C_6\left\|w_i\right\|_{L_2(G)}\cdot\left\|u_j\right\|_{W_2^s(G)}\leq\left\|w_i\right\|_{L_2(G)}\cdot\left\|u_j\right\|_{W_2^1(G)}^{s}\cdot\left\|u_j\right\|_{L_2(G)}^{1-s}=I_1, \tag{45}$$

where $s=3(p_0-1)/p_0$. We can conclude that

$$\left\|I_1\beta\right\|_{L_{q_0}(0,\infty)}^{q_0}\leq\int_0^{\infty}\beta^{q_0}\left\|w_i\right\|_{L_2(G)}^{q_0}\left\|u_j\right\|_{W_2^1(G)}^{sq_0}\left\|u_j\right\|_{L_2(G)}^{(1-s)q_0}dt\leq C\int_0^{\infty}\left\|\sqrt{\beta}w_i\right\|_{L_2(G)}^{q_0}\left\|\sqrt{\beta}u_j\right\|_{W_2^1(G)}^{sq_0}dt, \tag{46}$$

where $C=\left\|\sqrt{\beta}u_j\right\|_{L_\infty(0,\infty;L_2(G))}$. Applying the Hölder inequality with $q=\frac{2}{q_0}$, we infer

$$\left\|\beta I_1\right\|_{L_{q_0}(0,\infty)}\leq\left\|w_i\sqrt{\beta}\right\|_{L_2(Q)}^{q_0}\cdot[\int_0^{\infty}\left\|u_j\sqrt{\beta}\right\|_{W_2^1(G)}^{sq_0\frac{2}{2-q_0}}dt]^{\frac{2-q_0}{2q_0}}. \tag{47}$$

Note that $\frac{2sq_0}{2-q_0}=2$. In this case the previous inequality can be rewritten as

$$\|\beta I_1\|_{L_{q_0}(0,\infty)} \le C_8(M). \tag{48}$$

The expression

$$l(\vec\psi) = \sum_{i,j}\int_G w_i \psi_{jx_i} u_j dx = \left((\vec w,\nabla)\vec\psi,\vec u\right) \tag{49}$$

is a linear continuous functional over $\overset{\circ}{W}{}^1_{p_0'}(G)$. It follows from (44), (47) that

$$\|l(\vec\psi)\|_{W^{-1}_{p_0}(G)} = \sup_{\vec\psi\in \overset{\circ}{W}{}^1_{p_0'}(G)} \frac{|l(\vec\psi)|}{\|\vec\psi\|_{\overset{\circ}{W}{}^1_{p_0'}(G)}} \le C_9\|w_i\|_{L_2(G)}\cdot\|u_j\|^s_{W_2^1(G)}\cdot\|u_j\|^{1-s}_{L_2(G)}. \tag{50}$$

where $p_0' = p_0/(p_0-1)$. Using (48), we obtain that

$$\|l(\vec\psi)\beta\|_{L_{q_0}(0,\infty;W^{-1}_{p_0}(G))} \le C_{10}(M). \tag{51}$$

Denote

$$\left(\nabla p,\vec\psi\right) = l_1(\vec\psi), \nabla p\in L_{q_0}(0,\infty;L_{p_0}(G)). \tag{52}$$

This expression has a sense for functions $\vec\psi\in L_{q_0'}(0,\infty;L_{p_0'}(G))$ with a bounded support. Then we have

$$l_1(\vec\psi) \le \|\nabla p\|_{L_{p_0}(G)}\cdot\|\vec\psi\|_{L_{p_0'}(G)}. \tag{53}$$

In view of the estimates (37), (38), we derive

$$\left\|l_1(\vec\psi)\beta^\alpha\right\|_{L_{q_0}(0,\infty;W^{-1}_{p_0}(G))} \le \left\|\beta^\alpha\nabla p\right\|_{L_{q_0}(0,\infty;L_{p_0}(G))} \le C_{11}(M). \tag{54}$$

For integrals of the form

$$l_2(\vec\psi) = \left((\vec u,\nabla)\vec\psi,\vec u\right) = \sum_{i,j}\int_G \vec u_i\vec\psi_{jx_i}\vec u_j dxdt, l_3(\vec\psi) = \left((\vec u,\nabla)\vec\psi,\vec w\right),$$

we have the estimates (as in the proof of the estimate (53))

$$\|l_i(\vec\psi)\beta\|_{L_{q_0}(0,\infty;W^{-1}_{p_0}(G))} \le C_{12}(M), i=2,3. \tag{55}$$

Let $\{\varphi_i\}$ – be a basis for the subspace of the space $W_2^1(G)$, consisting of functions φ, satisfying the condition $\int_G \varphi dx = 0$. As vector functions $\{\vec\psi_i\}$, we choose the eigenfunctions of the problem

$$-\Delta\vec\psi = \lambda\vec\psi,\ \vec\psi|_\Gamma = 0,\ \vec\psi = (\psi_1,\psi_2,\psi_3)\in W_2^2(G)\cap\overset{\circ}{W}{}^1_2(G). \tag{56}$$

They form an orthonormalized basis for $L_2(G)$ (after normalization) and an orthogonal basis for the space $V = W_2^2(G)\cap\overset{\circ}{W}{}^1_2(G)$ if we take the expression $\langle\vec u,\vec v\rangle_V = (\Delta\vec u,\Delta\vec v)$ as a new inner product. Let P_N is orthoprojection in $L_2(G)$ on the subspace $V_N = Lin\{\vec\psi_1,\vec\psi_2,\ldots,\vec\psi_N\}$. It's obvious that $P_N\in L(V,V)$ and, in view of duality and selfadjointness, it allows an extension to a bounded operator of the class $L(V',V')$, where V' – dual space constructed by $L_2(G)$ and V as a completion of $L_2(G)$ with respect to the norm $\|u\|_{V'} = \sup_{v\in V}|\langle u,v\rangle_V|/\|v\|_V$. In particular, we have that $(u,P_Nv) = (P_Nu,v)$ for all $v\in V, u\in V'$. Note that $W_2^2(G)\cap\overset{\circ}{W}{}^1_2(G)\subset\overset{\circ}{W}{}^1_5(G)$ and the embedding is dense. This is a consequence of the embedding theorems. Since $V\subset\overset{\circ}{W}{}^1_5(G)$, we have that $W^{-1}_{5/4}(G)\subset V'$. Let λ_i be the corresponding eigenvalues.

We look for an approximate solution to the problem in the form

$$u_N = \sum_{i=1}^N c_i(t)\vec\psi_i(x),\ p_N = \sum_{i=1}^N \alpha_i(t)\varphi_i(x).$$

where $c_i(t)$ and $\alpha_i(t)$ are solutions to the system:

$$\left(\vec{u}_N-\vec{w}_N,\nabla\varphi_j\right)=0, a\left(\vec{u}_N,\vec{\psi}_j\right)=\left(\vec{f},\vec{\psi}_j\right),\ c_i(0)=\left(\vec{u}_0,\vec{\psi}_i\right),\ j=1,..,N. \quad (57)$$

The first equation of the system can be rewritten as

$$\left(\vec{u}_N-\frac{\tau\nabla p_N}{\rho}-\tau\left(\vec{u}_N\nabla\right)\vec{u}_N+f\tau,\nabla\varphi_i\right)=0. \quad (58)$$

We have that

$$\left(\frac{\tau\nabla p_N}{\rho},\nabla\varphi_i\right)=\frac{\tau}{\rho}\sum_{j=1}^{N}\alpha_j(t)\left(\nabla\varphi_j,\nabla\varphi_i\right),\det\left(\nabla\varphi_j,\nabla\varphi_i\right)\neq 0. \quad (59)$$

The last determinant is the Gram determinant and it does not vanish. Indeed, the following estimate holds:

$$\|\nabla p\|_{L_2(G)}\geq c_0\|p\|_{L_2(G)}\ \ \forall p\in W_2^1(G):\int_G p dx=0.$$

This inequality guarantees that an equivalent inner product $\langle u,v\rangle=\left(\nabla u,\nabla v\right)$ can be introduced in the required subspace of functions φ, which guarantees the claim. Let A be a matrix with elements $a_{ij}=\left(\nabla\varphi_i,\nabla\varphi_j\right)$. In this case the system (58) is rewritten in the form

$$\vec{\alpha}=\frac{\rho}{\tau}A^{-1}\begin{pmatrix}\left(\vec{u}_N-\tau\left(\vec{u}_N,\nabla\right)\vec{u}_N+\tau f,\varphi_1\right)\\ \left(\vec{u}_N-\tau\left(\vec{u}_N,\nabla\right)\vec{u}_N+\tau f,\varphi_N\right)\end{pmatrix}. \quad (60)$$

Substituting $\vec{\alpha}$ into the second system, we obtain a nonlinear system of ordinary differential equations for functions $c_i(t)$. The a priori estimate below guarantees that the Cauchy problem for this system has a solution on the entire interval $(0,\infty)$.

Further, we obtain a priori estimates for approximate solutions. Multiply the first and the second equation of the system (57) by α_i and c_i, respectively, and summarize the equalities over i. Then we obtain that

$$\left(\vec{u}_N-\vec{w}_N,\nabla p_N\right)=0,\ \ a\left(\vec{u}_N,\vec{u}_N\right)=\left(f,\vec{u}_N\right). \quad (61)$$

We have the above-proven estimates (19), (20) and, thus,

$$\left\|\vec{u}_N\sqrt{\beta}\right\|_{L_2(0,\infty;W_2^1(G))}+\left\|\operatorname{div}\vec{u}_N\sqrt{\beta}\right\|_{L_2(Q)}+\left\|\left(\frac{\nabla p_N}{\rho}+\left(\vec{u}_N,\nabla\right)\vec{u}_N\right)\sqrt{\beta}\right\|_{L_2(Q)}\leq C_1(M). \quad (62)$$

where $C_1(M)$ is some constant depending on M,μ,τ,

$$\left\|\vec{u}_N\sqrt{\beta}\right\|_{L_\infty(0,\infty;L_2(G))}\leq C_1(M). \quad (63)$$

The estimate has the same form because $\|P_N f\|_{L_2(G)}\leq\|f\|_{L_2(G)}$, $\|P_N u_0\|_{L_2(G)}\leq\|u_0\|_{L_2(G)}$, $u_N(0,x)=P_N u_0$. Take $p_0=q_0=5/4$ and fix the parameter $\alpha=\alpha_0$, satisfying the conditions from the statement of the theorem. As a consequence of (26), (28), (40), (42), we infer

$$\left\|w_N\sqrt{\beta}\right\|_{L_2(Q)}+\left\|\nabla p_N\beta^{\alpha_0}\right\|_{L_{5/4}(Q)}+\left\|\left(\vec{u}_N,\nabla\right)\vec{u}_N\beta\right\|_{L_{5/4}(Q)}\leq C_4(M). \quad (64)$$

Obtain an estimate for the derivative with respect to time of a solution. To this end, we rewrite the second equation of the system in the form

$$\left(\frac{\partial\vec{u}_N}{\partial t},\vec{\psi}\right)=\left(\left(\vec{u}_N-\vec{w}_N,\nabla\right)\vec{\psi},\vec{u}_N\right)-\frac{1}{\rho}\left(\nabla p_N,\vec{\psi}\right)+$$

$$\mu\left(\nabla\vec{u}_N,\nabla\vec{\psi}\right)-\mu\left(\operatorname{div}\vec{u}_N,\operatorname{div}\vec{\psi}\right)-\left(\left(\vec{u}_N,\nabla\right)\vec{\psi},\vec{w}_N\right)+\left(\vec{f},\vec{\psi}\right)=L_0(\vec{\psi}), \quad (65)$$

where $\vec{\psi}\in V_N$. It is easy to see that the expression $L_0(\vec{\psi})$ is a linear continuous functional over the space $\overset{\circ}{W}{}_5^1(G)$ in view of the estimates (53), (54), (55) (where the $\vec{u}$ is used instead of $\vec{u}_N$) and thereby also over the space V. Hence, there is $g_N(t)\in V'$ such that $L_0(\vec{\psi})=\left(g_N,\vec{\psi}\right)$ for all $\psi\in V$. The estimates (34), (53), (54), (55), (61)–(64) ensure that

$$\left\|g_N\beta^{\alpha_0}\right\|_{L_{5/4}(0,\infty;V')}\le\left\|g_N\beta^{\alpha_0}\right\|_{L_{5/4}(0,\infty;W_{5/4}^{-1}(G))}\le C_{12}\left(M\right),\qquad(66)$$

where C_{12} – some constant depending on M and independent of N. The equality (65) can be rewritten in the form

$$\vec{u}_{Nt}=P_Ng_N.$$

The previous estimates and boundedness of the operator P_N in V' imply that

$$\left\|\vec{u}_{Nt}\beta^{\alpha_0}\right\|_{L_{5/4}(0,\infty;V')}\le C_{12}\left(M\right),\qquad(67)$$

The sequence $\vec{u}_N$ is bounded in space endowed with the norm

$$\left\|\vec{u}\right\|=\left\|\vec{u}\beta^{\alpha_0}\right\|_{L_2(0,\infty;\overset{\circ}{W}{}_2^1(G))}+\left\|\vec{u}_t\beta^{\alpha_0}\right\|_{L_{5/4}(0,\infty;V')}\qquad(68)$$

and the estimates (62)–(64) are valid. Hence, there exists a subsequence $\vec{u}_{N_k}$ and function $\vec{u}\in L_{2,\sqrt{\beta}}(0,\infty;W_2^1(G))$ such that $\vec{u}_{N_k}\sqrt{\beta}\to\vec{u}\sqrt{\beta}$ in $L_2(0,\infty;W_2^1(G))$ weakly, $\vec{u}_{N_k x_i}\sqrt{\beta}\to\vec{u}_{x_i}\sqrt{\beta}$ weakly in $L_2\left(Q\right)$, $\vec{u}_{N_k t}\to\vec{u}_t$ weakly in $L_{5/4}(0,\infty;V')$, $\operatorname{div}\vec{u}_{N_k}\sqrt{\beta}\to\operatorname{div}\vec{u}\sqrt{\beta}$ weakly in $L_2\left(Q\right)$, $\vec{w}_{N_k}\sqrt{\beta}\to\vec{u}\sqrt{\beta}$ weakly in $L_2\left(Q\right)$, $p_{N_k}\beta^{\alpha_0}\to p\beta^{\alpha_0}$ weakly in $L_{5/4}\left(Q\right)$, $\beta^{\alpha_0}\nabla p_{N_k}\to\beta^{\alpha_0}\nabla p$ and $\beta(\vec{u}_{N_k},\nabla)\vec{u}_{N_k}\to\beta\vec{u}_1$ weakly in $L_{5/4}\left(Q\right)$, $\sqrt{\beta}\vec{u}_{N_k}\to\sqrt{\beta}\vec{u}$ weakly in $L_\infty(0,\infty;L_2(G))$. Demonstrate that

$$\vec{w}=\frac{\nabla p}{\rho}+\left(\vec{u},\nabla\right)\vec{u}-\vec{f},\ \vec{u}_1=\left(\vec{u},\nabla\right)\vec{u},\qquad(69)$$

Construct an increasing sequence $T_k\to\infty$ at $k\to\infty$. In view of the estimate (68), the subsequence $\vec{u}_{N_k}$ is bounded in the space endowed with the norm

$$\left\|\vec{u}\right\|_n=\left\|\vec{u}\right\|_{L_2(0,T_n;\overset{\circ}{W}{}_2^1(G))}+\left\|\vec{u}_t\beta^{\alpha_0}\right\|_{L_{5/4}(0,T_n;V')}.\qquad(70)$$

Next, we will use the compactness theorem (Theorem 5.1 of Chap. 1 in [22]). Note that the embedding $\overset{\circ}{W}{}_2^1(G)\subset L_2\left(G\right)$ is compact. By the compactness theorem, there exists a sequence $\vec{u}_{N_k}^1$ such that $\vec{u}_{N_k}^1\to\vec{u}$ strongly in $L_2(Q_{T_1})$, $Q_{T_1}=(0,T_1)\times G$ and almost everywhere in Q_{T_1}. Again using the compactness theorem, from the subsequence $\vec{u}_{N_k}^1$ we can select a subsequence $\vec{u}_{N_k}$ such that $\vec{u}_{N_k}^2\to\vec{u}$ strongly $L_2(Q_{T_2})$ and almost everywhere in Q_{T_2}. Repeating the arguments, we construct the family of subsequences $\vec{u}_{N_k}^i$ such that $\vec{u}_{N_k}^i\to\vec{u}$ strongly $L_2(Q_{T_i})$ and almost everywhere in Q_{T_i}. Now, define the subsequence $\vec{v}_k=\vec{u}_{N_k}^k$, which converges in $L_2(Q_{T_i})$ to $\vec{u}$ for all i and almost everywhere in Q. Fix i and take the function $\vec{\psi}\in L_\infty(Q)$ such that $supp\vec{\psi}\subset Q_{T_i}$. We have

$$\int_Q\beta\left(\left(\vec{v}_k,\nabla\right)\vec{v}_{N_k}-\left(\vec{u},\nabla\right)\vec{u}\right)\cdot\vec{\psi}dQ=\int_Q\beta\left(\left(\left(\vec{v}_k-\vec{u}\right),\nabla\right)\vec{v}_k+\left(\vec{u},\nabla\right)\left(\vec{u}-\vec{v}_k\right)\right)\cdot\vec{\psi}dQ.$$

The first integral is estimated as follows:

$$\left|\int_Q\beta\left(\left(\vec{v}_k-\vec{u}\right),\nabla\right)\vec{v}_k\cdot\vec{\psi}dQ\right|\le c_{13}\left\|\vec{v}_k-\vec{u}\right\|_{L_2(Q_{T_i})}\left\|\sqrt{\beta}\nabla\vec{v}_k\right\|_{L_2(Q_{T_i})}\to 0\ at\ k\to\infty.$$

Moreover, we have that

$$\int_Q (\vec{u},\nabla)(\vec{u}-\vec{v}_k)\cdot\vec{\psi}dQ \to 0 \; at \; k\to\infty,$$

due to weak convergence of $\beta\nabla\vec{v}_k$ to $\beta\nabla\vec{u}$ in $L_2(Q)$. Since the convergence takes place for all i and the set of functions with a bounded support of the class $L_\infty(Q)$ is dense in $L_{2,\beta}(0,\infty;L_2(G))$, we can conclude that $u_1=(\vec{u},\nabla)\vec{u}$ and thereby $\vec{w}=\frac{\nabla p}{\rho}+(\vec{u},\nabla)\vec{u}-\vec{f}$. A subsequence $\vec{v}_k$ coincides with some subsequence u_{N_k} for a suitable choice of the sequence N_k. Fix $T>0$ and take the set of functions $\alpha_i(t),c_i(t)\in C([0,\infty))$, such that $\sup\alpha_i,\sup c_i\subset[0,T]$, multiply the corresponding equalities (57) with $N=N_k$ by these functions, sum the result on i from 1 to n ($n\le N_k$) and integrate the obtained equations on t. As a result, we have

$$\int_0^\infty\left(\vec{u}_{N_k}-\vec{w}_{N_k},\nabla\varphi\right)dt=0,\; \int_0^\infty a\left(\vec{u}_{N_k},\vec{\psi}\right)dt=\int_0^\infty\left(\vec{f},\vec{\psi}\right)dt, \tag{71}$$

where $\vec{\psi}=\sum_{i=1}^n c_i\psi_i$ and $\varphi=\sum_{i=1}^n\alpha_i\varphi_i$. Let us consider successively all summands. First, we can pass to the limit in the first equality and obtain that

$$\int_0^\infty(\vec{u}-\vec{w},\nabla\varphi)dt=0,\vec{w}=\tau\left((\vec{u},\nabla)\vec{u}+\frac{1}{\rho}\nabla p-\vec{f}\right). \tag{72}$$

In the second equality, we consider only the nonlinear summands, since in the linear part the passage to the limit is realized due to the weak convergence. Consider the summand

$$J_{N_k}=\int_0^\infty\left(\left(\vec{u}_{N_k}-\vec{w}_{N_k},\nabla\right)\vec{\psi},\vec{u}_{N_k}\right)dt.$$

Demonstrate that $J_{N_k}\to J=\int_0^\infty\left((\vec{u}-\vec{w},\nabla)\vec{\psi},\vec{u}\right)dt$ as $k\to\infty$. Consider the difference

$$J_{N_k}-J=\int_0^\infty\left(\vec{u}_{N_k}-\vec{w}_{N_k},\nabla\right)\vec{\psi},\vec{u}_{N_k}-\vec{u}dt+\int_0^\infty\left(\left(\vec{u}_{N_k}-\vec{w}_{N_k},\nabla\right)\vec{\psi}\right),\vec{u}dt.$$

The second integral tends to zero due to weak convergence, and the first integral is estimated as follows:

$$\left|\int_0^\infty\left(\left(\vec{u}_{N_k}-\vec{w}_{N_k},\nabla\right)\vec{\psi},\vec{u}_{N_k}-\vec{u}\right)dt\right|\le c\left\|\vec{u}_{N_k}-\vec{u}\right\|_{L_2(Q_T)}\to 0 \text{ as} k\to\infty.$$

Similarly, we can show that $\int_0^\infty\left(\left(\vec{u}_{N_k},\nabla\right)\vec{\psi},\vec{w}_{N_k}\right)dt\to\int_0^\infty\left((\vec{u},\nabla)\vec{\psi},\vec{w}\right)dt$ at $k\to\infty$. Passing the limit as $k\to\infty$, we can conclude that

$$\int_0^\infty\left(\frac{\partial\vec{u}}{\partial t},\vec{\psi}\right)-\left((\vec{u}-\vec{w},\nabla)\vec{\psi},\vec{u}\right)+\frac{1}{\rho}(\nabla p,\vec{\psi})-\mu(\nabla\vec{u},\nabla\vec{\psi})+\mu(\operatorname{div}\vec{u},\operatorname{div}\vec{\psi})+\left((\vec{u},\nabla)\vec{\psi},\vec{w}\right)dt=\int_0^\infty\left(\vec{f},\vec{\psi}\right)dt.$$

In view of the choice the basis, we obtain that $\vec{u}$ is a generalized solution to the problem. Proof of the last statement of the theorem, i.e., inclusions $\nabla p,(u,\nabla u)u\in L_{q_0}(0,\infty;L_{p_0}(G))$, $\vec{u}_t\beta^\alpha\in L_{q_0}(0,\infty;W_{p_0}^{-1}(G))$ for any $p_0\in[1,3/2]$ and the corresponding parameters α was carried out in the first half of the proof.

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Received April 11, 2025

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Bulletin of the South Ural State University
Series “Mathematics. Mechanics. Physics”
2025, vol. 17, no. 3, pp. 13–27

УДК 517.954 **DOI: 10.14529/mmph250302**

ОБОБЩЕННАЯ РАЗРЕШИМОСТЬ НАЧАЛЬНО-КРАЕВЫХ ЗАДАЧ ДЛЯ КВАЗИГИДРОДИНАМИЧЕСКОЙ СИСТЕМЫ УРАВНЕНИЙ В ВЕСОВЫХ ПРОСТРАНСТВАХ СОБОЛЕВА

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Аннотация. В работе рассматривается аналог первой начально-краевой задачи для квазигидродинамической системы уравнений в случае слабосжимаемой жидкости в весовых пространствах Соболева. Система является эллиптико-параболической системой: первая ее часть представляет собой эллиптическое уравнение относительно градиента давления, а вторая представляет собой параболическую систему относительно вектора скорости. Неизвестные градиент давления и вектор скорости входят в главные части эллиптического уравнения и параболической системы. Стационарная часть системы не является равномерно эллиптической, что создает дополнительные трудности при исследовании задачи. Система была выведена Т.Г. Елизаровой и Б.Н. Четверушкиным путем осреднения известной кинетической модели. Первые варианты системы называются системой квазигазодинамических уравнений. Позднее Ю.В. Шеретовым на основе более общего уравнения состояния была получена еще одна модель, которая получила название «квазигидродинамическая система уравнений». Им же был проведен детальный анализ свойств этой системы. Однако ранее даже в линейном случае подробно не исследовались вопросы обобщенной разрешимости начально-краевых задач для таких систем, имеются только некоторые частные результаты. В данной статье будет предпринята попытка восполнить этот пробел. Доказывается обобщенная разрешимость системы в некоторых весовых классах, характеризующих поведение решений при $t\to\infty$. Доказательство основано на методе Галеркина и получаемых априорных оценках. Описано убывание (рост) решения в зависимости от убывания (роста) правой части системы. Убывание (рост) при $t\to\infty$ используемых весовых функций может быть как экспоненциальным, так и степенным.

Ключевые слова: начально-краевая задача; квазигидродинамическая система; априорные оценки; весовые функции; теорема существования.

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Поступила в редакцию 11 апреля 2025 г.

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