

OPTIMAL CONTROL OF SOLUTIONS TO THE SIXTH ORDER BOUSSINESQ EQUATION WITH TWO DISPERSIONS

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Abstract. This article investigates the Cauchy–Dirichlet and Showalter–Sidorov–Dirichlet problems for an inhomogeneous nonlinear sixth-order Boussinesq equation with two dispersions, which corresponds to an incomplete semilinear Sobolev-type equation of the second order in Banach spaces. Sobolev-type equations are distinguished by the non-invertibility of the operator at the highest time derivative. The problem posed models waves in a channel between lakes (or seas) with external wind forcing, internal damping, and convection. We use the Galerkin method adapted for this mathematical model to prove the existence and uniqueness of a nonlocal solution. The starting point for justifying the *-weak convergence of the Galerkin approximations is the formal representation of the solution as a Galerkin sum over the eigenfunctions of the operator $-\Delta$. The proof itself is based on the derived a priori estimates. Based on the proven theorem, we demonstrate the existence of a solution to the optimal control problem with a classical balance functional.

Keywords: sixth-order Boussinesq equation with two dispersions; optimal control problem; modified Galerkin method; degenerate equation.

Introduction

In a bounded domain Ω of the space R^n with a sufficiently smooth boundary $\partial\Omega$, and for $t \in [0, T]$, $T \in \mathbb{R}_+$ we consider the inhomogeneous nonlinear sixth-order Boussinesq equation with two dispersions [1]

$$(\lambda + \Delta^2 - \Delta)x_{tt} + (\alpha^2 \Delta^2 + \lambda \Delta)x - \Delta x^3 = f(s, t) + h(s, t), \quad (1)$$

with homogeneous Dirichlet boundary conditions

$$\Delta x(s, t) = x(s, t) = 0, \quad (s, t) \in \partial\Omega \times (0, T), \quad (2)$$

where $\alpha, \lambda \in \mathbb{R}$.

Equation (1) has numerous applications in various fields of natural science. For instance, in [1] its applicability to modeling waves in a channel between lakes (or seas) with external wind forcing, internal damping, and convection is demonstrated. In this model, the function $x = x(s, t)$ determines the wave height at point s at time t , $\alpha > 0$ is a constant depending on the water depth, and, λ is a constant characterizing the fluid properties. $f(s, t)$ is the external forcing function (e. g., wind), and $h(s, t)$ is a function characterizing the channel depth. In that work, wave damping in the channel is achieved by optimally determining the depth regulation function. For this purpose, using the maximum principle, the problem is reduced to an optimal control problem with initial-boundary conditions; as a result, without linearizing the nonlinear term, an optimal depth regulation function for the channel is obtained.

In [2], the Cauchy problem for a general class of nonlinear nonlocal wave equations arising in the framework of one-dimensional nonlocal elasticity theory is considered. Global existence of the solution to the Boussinesq equation is proven assuming sufficient smoothness of the initial data along with certain positivity conditions on the nonlinear term.

In [3], an equation analogous to (1) arises when studying nonlinear nonlocal wave equations in one-dimensional elasticity theory as an example. The global existence of solutions to the model is shown, assuming sufficient smoothness of the initial data together with some positivity conditions on the nonlinear term.

In [4], by combining methods from the theory of high-order Sobolev-type equations, monotone operator theory, and compactness methods, the existence of a unique solution to the initial-boundary value problem for the inhomogeneous complete semilinear Sobolev-type equation of the second order was proven for the case of self-adjoint and nonnegative definite operators in the linear part and an s -monotone, p -coercive, homogeneous nonlinear operator with a symmetric Fréchet derivative. The extendability of the local solution to an arbitrary time interval was demonstrated, which allowed for the formulation and solution of the optimal control problem for the nonlinear Boussinesq–Love equation.

The mathematical model (1), (2) can be written in operator form in Banach spaces as

$$L\ddot{x} + Mx + N(x) = u, \quad (3)$$

where $L = \lambda + \Delta^2 - \Delta$, $M = \alpha^2 \Delta^2 + \lambda \Delta$, $N(x) = -\Delta x^3$, $u = f(s, t) + h(s, t)$.

For equation (3), we investigate the Cauchy problem

$$x(0) = x_0, \quad \dot{x}(0) = x_1, \quad (4)$$

and the Showalter–Sidorov problem [5, 6]

$$P(x(0) - x_0) = 0, \quad P(\dot{x}(0) - x_1) = 0. \quad (5)$$

Here, P is a projector onto the coimage of the operator L . Problems (3), (4) and (3), (5) can be investigated using methods from the theory of Sobolev-type equations (the phase space method) [7]. If Banach spaces are chosen such that L is linear and continuous, M is linear, densely defined, and $N(x)$ is sufficiently smooth, then the theory of relatively p -bounded operators [8, 9] can be applied to show the existence and uniqueness of a local solution. For example, spaces such as

$$L, M, N: X \rightarrow Y; \quad X = \{u \in W_2^{k+4}(\Omega), u(x) = \Delta u(x) = 0, x \in \partial\Omega\}, \quad Y = \{W_2^k(\Omega)\}.$$

are suitable. In [10] a theorem on the existence of a unique local solution to problems (3), (4) and (3), (5), was proven, and it was also shown that in the case of monotonicity of the operator N , the phase space is a simple Banach manifold. However, this approach does not answer the question of how to find the solution. Therefore, we will also prove the existence and uniqueness theorem using the modified Galerkin method.

Let us formulate the optimal control problem. Let $\mathfrak{U}$ be the control space, and $\mathfrak{U}_{ad}$ a nonempty, convex, closed set of admissible controls. Then the problem for model (1), (2), (4) consists of minimizing the functional on $\mathfrak{U}_{ad}$:

$$J(x, u) \rightarrow \inf, \quad u \in \mathfrak{U}_{ad}. \quad (6)$$

The optimal control problem involves finding a compromise: on one hand, it is necessary to achieve a desired state of the system; on the other hand, to minimize the associated labor and energy costs. The initial formulation of such a problem for Sobolev-type models was presented in [10]. Subsequent research in this direction focused on studying semilinear Sobolev-type models of the first order. Optimal control methods acquire particular practical value when dealing with oscillatory processes. Typical examples include: damping of ship rolling, stabilization of a crane boom, creation of effective vibration protection systems, and other similar problems. The significance of solving such problems has been repeatedly emphasized in the scientific literature [11–13].

Besides the introduction and the list of references, the article consists of 3 sections. The first section is devoted to the phase space method. The second section investigates the Cauchy and Showalter–Sidorov problems for the inhomogeneous equation. The third section considers the optimal control problem.

1. Phase Space Method

The abstract problem (3), (4) was previously investigated using methods from the theory of relatively bounded operators [9]. Its main tenets are presented below.

Let $\mathfrak{X}$, $\mathfrak{Y}$ be Banach spaces, let the operator $L \in \mathcal{L}(\mathfrak{X}; \mathfrak{Y})$ (i. e., linear and continuous), and let $M \in Cl(X; Y)$ (linear, closed, and densely defined).

The set

$$\rho^L(M) = \left\{ \mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{Y}; \mathfrak{X}) \right\}$$

is called the resolvent set of the operator M with respect to the operator L (or the L -resolvent set of M). The set $\mathbb{C} \setminus \rho^L(M) = \sigma^L(M)$ is called the spectrum of M with respect to L (or the L -spectrum of M).

The operator functions of a complex variable $(\mu L - M)^{-1}$, $R_\mu^L = (\mu L - M)^{-1}L$, $L_\mu^L = L(\mu L - M)^{-1}$ defined on the L -resolvent set of M are called the L -resolvent, the right L -resolvent, and the left L -resolvent of M , respectively.

The operator M is called (L, σ) -bounded if its L -spectrum lies in a bounded set. Let M be (L, σ) -bounded and let Γ be a contour bounding the L -spectrum of M . Then the operators

$$P = \frac{1}{2\pi i} \int_{\Gamma} R_\lambda^L(M) d\lambda, \quad Q = \frac{1}{2\pi i} \int_{\Gamma} L_\lambda^L(M) d\lambda$$

are projectors in the spaces $\mathfrak{X}$ and $\mathfrak{Y}$, respectively. Let $\ker P = \mathfrak{X}^0$, $\operatorname{im} P = \mathfrak{X}^1$, $\ker Q = \mathfrak{Y}^0$, $\operatorname{im} Q = \mathfrak{Y}^1$.

The projectors P and Q split the spaces $\mathfrak{X}$, $\mathfrak{Y}$ into direct sums: $\mathfrak{X} = \mathfrak{X}^0 \oplus \mathfrak{X}^1$ and $\mathfrak{Y} = \mathfrak{Y}^0 \oplus \mathfrak{Y}^1$ such that the actions of the operators L and M make sense as operators $H = M_0^{-1}L_0 \in \mathcal{L}(\mathfrak{X}^0)$, $S = L_1^{-1}M_1 \in \mathcal{L}(\mathfrak{X}^1)$. The operator M is called (L, p) -bounded if $M(L, \sigma)$ -bounded and the operator H is nilpotent of degree p .

Let $\mathfrak{P}$ be a Banach manifold without boundary modeled by the space $\mathfrak{X}$. We say that a pair (x, y) belongs to the tangent bundle of the set $\mathfrak{P}$ if x belongs to $\mathfrak{P}$ and the pair (x, y) belongs to the tangent space $T_x \mathfrak{P}$ at the point x i. e.

$$(x, y) \in T\mathfrak{P} \Leftrightarrow x \in \mathfrak{P} \wedge (x, y) \in T_x \mathfrak{P}.$$

The set $\mathfrak{P}$ is called the phase space of equation (3) if

- 1) for any $(x_0, x_1) \in T\mathfrak{P}$ there exists a unique solution to problem (3), (4);
- 2) any solution $x = x(t)$ of equation (3) lies in $\mathfrak{P}$ as a trajectory.

Let $\ker L \neq \{0\}$ and let M $(L, 0)$ -bounded. Then, by the splitting theorem [11], equation (3) can be reduced to an equivalent system of equations

$$\begin{cases} (\mathbb{I} - Q)(M + N)(x) = (\mathbb{I} - Q)u, \\ \dot{x}^1 = -L_1^{-1}Q(M + N)(x) + L_1^{-1}Qu, \end{cases}$$

where $x^1 = Px$, $N = \Delta x^3$.

Let $(\mathbb{I} - Q)u$ be independent of t . Then the local phase manifold of equation (3) is the set $\mathfrak{P} = \{x \in \mathfrak{X} : (\mathbb{I} - Q)(M + N)(x) = (\mathbb{I} - Q)u\}$. The condition that the initial data belongs to the tangent bundle of the phase space is a necessary condition for the solvability of the Cauchy problem. Thus, in [7], the existence of a unique local solution was proven. It is easy to show that the mathematical model (1), (2) with initial conditions (3) satisfies the conditions of the abstract theorem when $\lambda \cdot \alpha^{-2} < 1$.

Furthermore, we will need an auxiliary lemma.

Lemma 1 [13] If $f \in L^p(0, T; \mathfrak{X})$ and $\dot{f} \in L^p(0, T; \mathfrak{X})$, then, after possibly being redefined on a set of measure zero from the interval $(0, T)$, f is a continuous mapping from $[0, T]$ into $\mathfrak{X}$.

2. Investigation of the Cauchy and Showalter–Sidorov Problems for the Inhomogeneous Equation

We investigate problem (1), (2), (4) using the operator form for convenience. We introduce the function spaces: $H = W_2^{-1}(\Omega)$ is a real, separable Hilbert space with scalar product

$$\langle x, y \rangle = \int_{\Omega} x \tilde{y} ds, \quad \forall x, y \in W_2^{-1}(\Omega),$$

where $\tilde{y}$ is the generalized solution of the Dirichlet problem (2) for the equation $-\Delta\tilde{y} = y$ in Ω ; the Sobolev space with compact support $\mathfrak{X} = \{x \in H^2(\Omega) : \Delta x(s, t) = x(s, t) = 0, (s, t) \in \partial\Omega \times (0, T)\}$ – a real, separable Hilbert space; the Lebesgue space $L^4(\Omega)$. Note that $\mathfrak{X}$ and L^4 form dual pairs of reflexive separable Banach spaces $(\mathfrak{X}, \mathfrak{X}^*)$ and $(L^4, (L^4)^*)$ with respect to the duality $\langle \cdot, \cdot \rangle$.

Let λ_k be the eigenvalues of the homogeneous Dirichlet problem in the domain Ω for the operator $\Delta^2 - \Delta$, numbered in non-decreasing order counting multiplicity, and let φ_k be the corresponding eigenfunctions, orthonormal with respect to the scalar product in H . Furthermore, the linear span $\text{span}\{\varphi_1, \varphi_2, \dots, \varphi_m\}$ is dense in $\mathfrak{X}$ as $m \rightarrow \infty$.

We define the operator $L : \mathfrak{X} \rightarrow \mathfrak{X}^*$ by the formula

$$\langle Lx, y \rangle = \int_{\Omega} (\lambda x \tilde{y} - \nabla x \nabla y + xy) ds.$$

For $\lambda \geq \lambda_1$ $L \in \mathcal{L}(\mathfrak{X}, \mathfrak{X}^*)$ is self-adjoint, nonnegative definite, and Fredholm.

We define the operator $M : \mathfrak{X} \rightarrow \mathfrak{X}^*$ by the formula

$$\langle Mx, y \rangle = \int_{\Omega} (-\alpha^2 \nabla x \nabla y + \lambda xy) ds.$$

Then, for $\lambda \cdot \alpha^{-2} < 1$, $M \in \mathcal{L}(\mathfrak{X}, \mathfrak{X}^*)$ is self-adjoint and nonnegative definite.

We define the operator $N(x) : L^4 \rightarrow (L^4)^*$ by the formula

$$\langle N(x), y \rangle = \int_{\Omega} x^3 y ds.$$

Its Frechet derivative

$$|\langle N'_x(y), w \rangle| = 3 \left| \int_{\Omega} x^2 y w ds \right| \leq \text{const} \|x\|_{L^2} \|y\|_{L^4} \|w\|_{L^4}$$

is symmetric and bounded by Holder's inequality.

The operator N is monotone:

$$\langle N'_x(y), y \rangle = 3 \int_{\Omega} x^2 y^2 ds \geq 0$$

and 4-coercive

$$\langle N(x), x \rangle = \int_{\Omega} x^3 x ds = \|x\|_{L^4}^4, \quad \langle N(x), y \rangle = \int_{\Omega} x^3 y ds = \|x\|_{L^4}^3 \|y\|_{L^4} = \|x\|_{\frac{4}{3}} \|y\|_{L^4}.$$

Furthermore, the operator N , due to the properties mentioned above, satisfies the equality

$$\frac{d}{dt} \langle N(x), x \rangle = 4 \langle N(x), \dot{x} \rangle.$$

We define the distribution spaces (functions with values in a Banach space) $L^\infty(0, T; \mathfrak{X} \cap L^4)$ and $L^\infty(0, T; \text{coim } L)$, $\mathfrak{X} = \ker L \oplus \text{coim } L$, which are dual to $L^1(0, T; \mathfrak{X}^* \cup (L^4)^*)$ and $L^1(0, T; \mathfrak{X}^*)$, respectively, by the Dunford–Pettis theorem.

Theorem 1. Let $\lambda \geq \lambda_1, \lambda \cdot \alpha^{-2} < 1$ and $u \in L^2(0, T; (L^4)^*)$. Then for any $(x_0, x_1) \in T_{x_0} \mathfrak{P}$, such that $x_0 \in \mathfrak{X} \cap L^4$, $x_1 \in \text{coim } L$ there exists a unique solution $x = x(s, t)$ to problem (1), (2), (4) such that $x \in L^\infty(0, T; \mathfrak{X} \cap L^4)$ and $\dot{x} \in L^\infty(0, T; \text{coim } L \cap \mathfrak{X})$.

We seek a solution to problem (3), (4) in the form of a Galerkin approximation

$$x^m(t) = \sum_{k=1}^m a_k^m(t) \varphi_k.$$

The coefficients $a_k^m(t)$ are found from the system of algebraic-differential equations

$$\langle L\ddot{x}^m, \dot{x} \rangle + \langle Mx^m, \dot{x} \rangle + \langle x^3, \dot{x} \rangle = \langle u, \dot{x} \rangle, \quad 1 \leq k \leq m. \quad (7)$$

$$\langle x^m(0), \varphi_k \rangle = \langle x_0, \varphi_k \rangle = \beta_k^m, \quad \langle \dot{x}^m(0), \varphi_k \rangle = \langle x_1, \varphi_k \rangle = \gamma_k^m, \quad 1 \leq k \leq m,$$

with $x_0^m = \sum_{k=1}^m \beta_k^m \varphi_k \rightarrow x_0$ in $\mathfrak{X}$ as $m \rightarrow \infty$, and $x_1^m = \sum_{k=1}^m \gamma_k^m \varphi_k \rightarrow x_1$ and $\mathfrak{X}$ as $m \rightarrow \infty$.

By classical results, there exists a unique local solution $x^m = x^m(s, t)$, $t \in [0, t^m]$.

We show that the norms of the local solutions of the unknown function and its derivative are bounded in the given spaces on some time interval $[0, T]$. Multiply equation (7) by $\dot{a}_k^m(t)$ ($1 \leq k \leq m$) and sum over k from 1 to m . We obtain

$$\langle L\dot{x}^m, \dot{x}^m \rangle + \langle Mx^m, \dot{x}^m \rangle + \langle N(x^m), \dot{x}^m \rangle = \langle u, \dot{x}^m \rangle. \quad (8)$$

We introduce in the space $\text{coim } L \cap \mathfrak{X}$ the norm $|\dot{x}|^2 = \langle L\dot{x}, \dot{x} \rangle$. By Courant's principle, this norm is equivalent to the norm induced from $\mathfrak{X}$ for $x \in \text{coim } L$.

Using the self-adjointness of the operators L and M , we get $2\langle L\dot{x}^m, \dot{x}^m \rangle = \frac{d}{dt} \langle Lx^m, x^m \rangle$, $2\langle Mx^m, \dot{x}^m \rangle = \frac{d}{dt} \langle Mx^m, x^m \rangle$.

Then equation (8) takes the form

$$\frac{d}{dt} \left[|\dot{x}^m|^2 + \langle Mx^m, x^m \rangle + \frac{1}{2} \langle N(x^m), x^m \rangle \right] = 2\langle u, x^m \rangle.$$

Integrate over $[0, t]$, $t \leq t_m$

$$|\dot{x}^m|^2 + \langle Mx^m, x^m \rangle + \frac{1}{2} \langle N(x^m), x^m \rangle = 2 \int_0^t \langle u, \dot{x}^m \rangle ds + |\dot{x}_1^m|^2 + \langle Mx_0^m, x_0^m \rangle + \frac{1}{2} \langle N(x_0^m), x_0^m \rangle \quad (9)$$

$$|\dot{x}^m|^2 + \langle Mx^m, x^m \rangle + C_N \frac{1}{2} \|x^m\|_{(L^4)}^4 \leq$$

$$2 \int_0^t \|u\|_{(L^4)^*} \|\dot{x}^m\|_{L^4} ds + |\dot{x}_1^m|^2 + \langle Mx_0^m, x_0^m \rangle + C^N \frac{1}{2} \|N(x_0^m)\|_{(L^4)^*} \leq \quad (10)$$

$$\leq \int_0^t \|u\|_{(L^4)^*}^2 ds + \int_0^t \|\dot{x}^m\|_{L^4}^2 ds + |\dot{x}_1^m|^2 + \langle Mx_0^m, x_0^m \rangle + C^N \frac{1}{2} \|N(x_0^m)\|_{(L^4)^*}.$$

From this, in particular, it follows that

$$|\dot{x}^m|^2 \leq C + \int_0^t \|\dot{x}^m\|_{L^4}^2 ds \leq C + \int_0^t \|\dot{x}^m\|_{\mathfrak{X}}^2 ds \leq C + \int_0^t |\dot{x}^m|^2 ds,$$

and by Gronwall's inequality, we get

$$|\dot{x}^m|^2 \leq Ce^t \leq C, \quad t \in [0, T].$$

Since the right-hand side of inequality (10) is bounded, the following inequality holds:

$$|\dot{x}^m|^2 + \langle Mx^m, x^m \rangle + C_N \frac{1}{2} \|x^m\|_{L^4}^p \leq C.$$

The constant C is independent of t_m , and therefore $t_m = T$.

From the obtained a priori estimate, it is evident that the sequences x^m and $\dot{x}^m$ are bounded in the spaces $L^\infty(0, T; \mathfrak{X} \cap L^4)$ and $L^\infty(0, T; \text{coim } L)$, respectively, which are dual spaces to the separable Ba-

nach spaces $L^1\left(0, T; \mathfrak{X}^* \cup (L^4)^*\right)$ and $L^1\left(0, T; \mathfrak{X}^*\right)$. Therefore, we can select *-weakly convergent subsequences x^{m_l} and $\dot{x}^{m_l}$ such that

$$x^{m_l} \rightarrow x \text{ *-weakly in } L^\infty\left(0, T; \mathfrak{X} \cap L^4\right),$$

$$\dot{x}^{m_l} \rightarrow \dot{x} \text{ *-weakly in } L^\infty\left(0, T; \text{coim } L\right).$$

Moreover, $\dot{x}^{m_l}$ is understood as a generalized derivative in the space of distributions. Since the operator N is coercive:

$$\langle N(x^m), x^m \rangle \leq \|N(x^m)\|_{(L^4)^*} \|x^m\|_{L^4} \leq C^N \|x^m\|_{L^4}^3 \|x^m\|_{L^4} = C^N \|x^m\|_{L^4}^4,$$

and, consequently, $N(x^m)$ are bounded in the space $\left(L^4\left(0, T; \mathfrak{X}^* \cup (L^4)^*\right)\right)^*$, then $N(x^{m_l}) \rightarrow g$ - weakly in $\left(L^4\left(0, T; \mathfrak{X}^* \cup (L^4)^*\right)\right)^*$.

We show that $N(x) = g$. From the monotonicity of N it follows that

$$X^{m_l} = \langle N(x^{m_l}) - N(z), x^{m_l} - z \rangle \geq 0, \quad \forall z \in L^4\left(0, T; \mathfrak{X} \cap L^4\right).$$

From formula (9)

$$\begin{aligned} X^{m_l} = & 2 \int_0^t \langle u, \dot{x}^{m_l} \rangle ds + |x_1^{m_l}|^2 - |\dot{x}^{m_l}|^2 + \langle Mx_0^{m_l}, x_0^{m_l} \rangle - \langle Mx^{m_l}, x^{m_l} \rangle + \\ & + \frac{1}{2} \langle N(x_0^{m_l}), x_0^{m_l} \rangle - \frac{1}{2} \langle N(x^{m_l}), z \rangle - \langle N(z), x^{m_l} - z \rangle. \end{aligned}$$

By the properties of weakly convergent sequences, we have $\liminf_{m_l \rightarrow \infty} |x^{m_l}(t)|^2 \geq |x(t)|^2$, therefore

$$\limsup_{m_l \rightarrow \infty} X^{m_l} \leq \langle g, x \rangle - \langle g, z \rangle - \langle N(z), x - z \rangle.$$

Then we get

$$\langle g, x \rangle - \langle g, z \rangle - \langle N(z), x - z \rangle \geq 0.$$

Set $z = x - hw$, $h > 0$, $w \in L^4\left(0, T; \mathfrak{X} \cap L^4\right)$, Then

$$\langle g - N(x - hw), hw \rangle \geq 0, \quad \langle g - N(x - hw), w \rangle \geq 0.$$

Letting $h \rightarrow 0$, by the continuity of N and Lebesgue's dominated convergence theorem, we obtain

$$\langle g - N(x), w \rangle \geq 0$$

Since w is arbitrary, we have $g = N(x)$.

Now we can pass termwise to the limit in equality (7) as $m_l \rightarrow \infty$. Let k be fixed and $m_l > k$. We get

$$\langle L\ddot{x}^{m_l}, \varphi_k \rangle + \langle Mx^{m_l}, \varphi_k \rangle + \langle N(x^{m_l}), \varphi_k \rangle = \langle u, \varphi_k \rangle. \quad (11)$$

Thus, from (11) we deduce

$$\frac{d^2}{dt^2} \langle Lx, \varphi_k \rangle + \langle Mx, \varphi_k \rangle + \langle N(x), \varphi_k \rangle = \langle u, \varphi_k \rangle.$$

Due to the density of the system of functions $\{\varphi_k\}_{k=1}^m$ in $\mathfrak{X}$ as $m \rightarrow \infty$, and the arbitrary choice of φ_k , the equality holds for any $v \in \mathfrak{X}$

$$\frac{d^2}{dt^2} \langle Lx, v \rangle + \langle Mx, v \rangle + \langle N(x), v \rangle = \langle u, v \rangle.$$

From the expansion of the initial functions into series $x^{m_l}(0) = x_l^0 \rightarrow x_0$ weakly in $\mathfrak{X}$, and $x^{m_l}(0) \rightarrow x(0)$ in $\mathfrak{X}$, hence, $x(0) = x_0$.

From the obtained a priori estimates

$$\langle \dot{x}^{m_l}, \varphi_k \rangle \rightarrow \langle \dot{x}, \varphi_k \rangle \text{ *-weakly in } L^\infty(0, T)$$

and, therefore, considering Lemma 1, we get

$$\dot{x}^{m_l}(0), \varphi_k \rightarrow \dot{x}(t), \varphi_k|_{t=0} = \dot{x}(0), \varphi_k.$$

On the other hand, from the expansion of the initial functions

$$\langle \dot{x}^{m_l}(0), \varphi_k \rangle \rightarrow \langle x_1, \varphi_k \rangle.$$

Thus,

$$\langle \dot{x}(0), \varphi_k \rangle = \langle x_1, \varphi_k \rangle, \quad \forall k.$$

Uniqueness is proven by contradiction $x = x(s, t)$ satisfies the equation and the initial conditions, i.e., it is a solution to (1), (2), (4).

Uniqueness is proven by contradiction. Let $x_1(s, t)$ and $x_2(s, t)$ be two distinct solutions to problem (1), (2), (4). Denote $w(s, t) = x_1(s, t) - x_2(s, t)$. Repeating the arguments above, we arrive at the inequality

$$|\dot{w}^m|^2 \leq \int_0^t \dot{w}_{L^4}^{m2} ds \leq \int_0^t \dot{w}_{\mathfrak{X}}^{m2} ds \leq \int_0^t |\dot{w}^m|^2 ds,$$

and by Gronwall's inequality,

$$|\dot{w}^m|^2 \leq 0, \quad t \in [0, T].$$

Taking into account zero initial values, we get $w(s, t) = 0$.

Repeating the arguments of the proof above, it is easy to prove the following theorem on the existence of a solution to the Showalter–Sidorov problem.

Theorem 2. Let $\lambda \geq \lambda_1, \lambda \cdot \alpha^{-2} < 1$ and $u \in L^2\left(0, T; (L^4)^*\right)$. Then for any $x_0 \in \mathfrak{X} \cap L^4$, $x_1 \in \text{coim} L$

there exists a solution $x = x(s, t)$ to problem (1), (2), (5) such that $x \in L^\infty(0, T; \mathfrak{X} \cap L^4)$ and $\dot{x} \in L^\infty(0, T; \mathfrak{X})$.

3. Optimal Control Problem

To investigate the optimal control problem (1), (2), (4), (6) we construct the corresponding space $\mathfrak{U} = L^2\left(0, T; (L^4)^*\right)$, and then introduce in it a nonempty, closed, and convex set of admissible controls

$\mathfrak{U}_{ad}$. Let's build a space $\mathfrak{X}_1 = \{x | x \in L^\infty(0, T; \mathfrak{X} \cap L^4), \dot{x} \in L^\infty(0, T; \text{coim} L \cap \mathfrak{X})\}$.

A pair $(\tilde{x}, \tilde{u}) \in \mathfrak{X}_1 \times \mathfrak{U}_{ad}$ is called a solution to the optimal control problem if

$$J(\tilde{x}, \tilde{u}) = \inf_{(x, u)} J(x, u),$$

where pairs $(x, u) \in \mathfrak{X}_1 \times \mathfrak{U}_{ad}$ satisfy problem (1), (2), (4). The function $\tilde{u}$ is called the optimal control. An admissible element of problem (1), (2), (4), (6) is a pair $(x, u) \in \mathfrak{X}_1 \times \mathfrak{U}_{ad}$, satisfying problem (1), (2), (4) for which $J(x, u) < +\infty$. Since the set $\mathfrak{U}_{ad} \neq \emptyset$, for any $u \in \mathfrak{U}_{ad} \subset \mathfrak{U}$ by Theorem 1, there exists a unique solution $x = x(u)$ to problem (1), (2), (4).

Let us formulate theorems on the existence of optimal control for the Cauchy and Showalter–Sidorov problems.

Theorem 3. Let $\lambda \geq \lambda_1$ and $\lambda \cdot \alpha^{-2} < 1$ hold. Then for any $(x_0, x_1) \in T\mathfrak{P}$, $T \in \mathbb{R}_+$, there exists a solution to problem (1), (2), (4), (6).

From the theorem on the existence of a unique solution to problem (1), (2), (4), it follows that the operator

$$\left(L \frac{d}{dt^2} + M + N(\cdot) \right): \mathfrak{X}_1 \rightarrow \mathfrak{U}$$

is a homeomorphism. We define the cost functional used in (6) as

$$J(x, u) = J(u) = \beta \left(x(t) - z(t)_{L^4(0, T; \mathfrak{X} \cap L^4)}^4 + \dot{x}(t) - \dot{z}(t)_{L^2(0, T; \text{coim} L)}^2 \right) + (1 - \beta) u(t)_{\left(L^2(0, T; L^4 \cap \mathfrak{X}) \right)^*}^2, \quad (12)$$

where $\beta \in (0, 1)$ is a weight coefficient. Let $\{u_m\} \subset \mathfrak{U}_{ad}$ be a sequence such that

$$\lim_{m \rightarrow +\infty} J(u_m) = \inf_{u \in \mathfrak{U}_{ad}} J(u), \quad (13)$$

then from (13) it follows that

$$\|u(t)\|_{L^2\left(0, T; \left(L^4\right)^*\right)}^2 \leq C \quad (14)$$

for all $m \in \mathbb{N}$. Since functional (12) is not strictly convex, the solution to the optimal control problem may not be unique. According to (14), from the sequence $\{u_m\}$ we can extract a weakly convergent subsequence (which, for convenience, we keep denoted as $\{u_m\}$, such that $u_m \rightarrow \tilde{u}$. By Mazur's theorem, $\tilde{u} \in \mathfrak{U}_{ad}$. Let $x_m = x(u_m)$ be the weak generalized solution of the equation

$$L\ddot{x}_m + Mx_m + N(x) = u_m.$$

A priori estimates are derived as in the proof of Theorem 1.

Now we can pass termwise to the limit in equality (7) as $m \rightarrow \infty$:

$$L\ddot{\tilde{x}} + M\tilde{x} + N(\tilde{x}) = \tilde{u}.$$

Thus, $\tilde{x} = \tilde{x}(\tilde{u})$ и $\liminf_{m \rightarrow \infty} J(x_m) \leq J(\tilde{x})$. Consequently, $\tilde{x}$ is a solution to the optimal control problem (1), (2), (4), (6).

Repeating the arguments of the proof above, it is easy to prove the theorem on the existence of a solution to the optimal control problem with solutions of the Showalter–Sidorov problem.

Theorem 4. Let $\lambda \geq \lambda_1$ and $\lambda \cdot \alpha^{-2} < 1$ hold. For any $x_0, x_1 \in \mathfrak{X}$, $T \in \mathbb{R}_+$ there exists a solution to the optimal control problem (1), (2), (5), (6).

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ОПТИМАЛЬНОЕ УПРАВЛЕНИЕ РЕШЕНИЯМИ УРАВНЕНИЯ БУССИНЕСКА ШЕСТОГО ПОРЯДКА С ДВУМЯ ДИСПЕРСИЯМИ

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Аннотация. Исследуется задача Коши–Дирихле и Шоуолтера–Сидорова–Дирихле для неоднородного нелинейного уравнения Буссинеска шестого порядка с двумя дисперсиями, отвечающего неполному полулинейному уравнению соболевского типа второго порядка в банаховых пространствах. Особенность уравнений соболевского типа состоит в необратимости оператора при старшей производной по времени. Поставленная задача моделирует волны в канале между озерами (морями) с внешним ветровым воздействием, внутренним затуханием и конвекцией. Методом Галеркина, адаптированным для указанной математической модели, доказано существование и единственность нелокального решения. Исходным пунктом для обоснования *-слабой сходимости Галёркинских аппроксимаций выступает формальное представление решения в виде Галёркинской суммы по собственным функциям оператора $-\Delta$, а само доказательство базируется на полученных априорных оценках. Опираясь на доказанную теорему, удалось доказать существование решения задачи оптимального управления с классическим балансовым функционалом.

Ключевые слова: уравнение Буссинеска с двумя дисперсиями шестого порядка; задача оптимального управления; модифицированный метод Галеркина; вырожденное уравнение.

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