INVERSE PROBLEMS OF RECOVERING THE BOUNDARY DATA WITH INTEGRAL OVERDETERMINATION CONDITIONS

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In the present article we examine an inverse problem of recovering unknown functions being part of the Dirichlet boundary condition together solving an initial boundary problem for a parabolic second order equation. Such problems on recovering the boundary data arise in various tasks of mathematical physics: control of heat exchange prosesses and design of thermal protection systems, diagnostics and identification of heat transfer in supersonic heterogeneous flows, identification and modeling of heat transfer in heat-shielding materials and coatings, modeling of properties and heat regimes of reusable heat protection of spacecrafts, study of composite materials, etc. As the overdetrermination conditions we take the integrals of a solution over the spatial domain with weights. The problem is reduced to an operator equation of the Volterra-type. The existence and uniqueness theorem for solutions to this inverse problem is established in Sobolev spaces. A solution is regular, i. e., all generalized derivatives occuring into the equation exists and are summable to some power. The proof relies on the fixed point theorem and bootstrap arguments. Stability estimates for solutions are also given. The solvability conditions are close to necessary conditions.

Keywords: inverse problem; parabolic equation; boundary and initial condition; Sobolev space; existence and uniqueness theorem; solvability.

Introduction

We consider the parabolic equation

$$Lu = u_t - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(t,x)u_{x_j} + \sum_{i=1}^{n} a_i(t,x)u_{x_i} + a_0(t,x)u = f, \tag{1}$$

where $x \in G \subset \mathbb{R}^n$ is a bounded domain with boundary Γ of the class C^2 (see the definition, for instance, in [1, Sect. 1]), $t \in (0,T)$. Put $Q = (0,T) \times G$ and $S = (0,T) \times \Gamma$. The equation (1) is furnished with the following initial and boundary conditions:

$$u|_S = g, \ u|_{t=0} = u_0(x). \tag{2}$$

Put $\frac{\partial u}{\partial N} = \sum_{i,j=1}^{n} a_{ij}(t,x)u_{x_j}(t,x)v_i$, where $v = (v_1, v_2, \ldots, v_n)$ is the outward unit normal to S. The inverse problem is to find a solution u to the problem (1)–(2) and a function g of the form $g = \sum_{i=1}^{m} q_i(t)\Phi_i(t,x)$, where the vector $\vec{q} = (q_1, q_2, \ldots, q_m)$ is unknown, with the use of the overdetermiantion conditions

$$\int_G u(x,t)\varphi_k(x)dx = \psi_k(t), k = 1,2,\ldots,m. \tag{3}$$

Inverse problems of recovering boundary regimes, in particular, the convective heat exchange problems are conventional (see, for instance, [2–11]). They arise in different problems of mathematical physics such as the problems of control of heat exchange prosesses and design of thermal protection systems, diagnostics and identification of heat transfer in supersonic heterogeneous flows, identification and modeling of heat transfer in heat-shielding materials and coverings, modeling of properties and heat regimes of reusable heat protection of spacecrafts, the study of composite materials, etc. Mathematical models describing these prosesses and the corresponding inverse problems in both one-dimensional and multidimensional cases are described, for example, in [2]. The essential attention here is paid to numerical methods of solving inverse problems in question and some uniqueness theorems together stability estimates for solutions. We refer also to the monograph [3] mainly devoted to numerical

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methods of determining a solution, where in the one-dimensional case different inverse problems for parabolic equations and problems of recovering the boundary regimes as well are studied. The overdetrermination conditions are the values of a solution at some points lying inside the spatial domain. These problems are studied in different settings in dependence on the type of the ovedetermination conditions. It is often the case when these problems are ill-posed in the Hadamard sense. In particular, it is true in the case of the overdetermination conditions in the form of values of a solution at separate points or on some surfaces lying in the spatial domain (see [2]). At the present article we examine the problems with overdetermination conditions in the form of some integrals with weights of a solution over a spatial domain. Note that these conditions arise in applications and they are often used in literature. Inverse problems of recovering the right-hand side or coefficients of an equation with integral ovedetermination conditions are studied in the articles [12–18], the monographs [19, 20], and some other articles. In particular, the existence and uniqueness theorem of a generalized solution to the problem (1)–(3) (from the class $u \in W_2^{0,1}(Q)$) in the case of $m = 1$ and the Neumann boundary condition was obtained in [9] and a similar result for a heat-and-mass transfer system including the Navier–Stokes system and a parabolic equation for the concentration of an admixture was obtained in [10]. The article [11] is devoted to a regular solvability ($u \in W_2^{1,2}(Q)$) in the case of $m = 1$ and the Robin boundary conditions. The case of the Dirichlet boundary condition happens to be more complicated than the case of the Neumann (Robin) boundary conditions and was not studied before. The present article is devoted to this case. Under some conditions on the data we prove well-poseness of this problem. The article in some sense is an extension of [21], where the Robin boundary conditions are treated. Some our auxiliary statements are taken from this article.

Preliminaries

Let E be a Banach space. Denote by $L_p(G;E)$ (G is a domain in $\mathbb{R}^n$) the space of strongly measurable functions defined on G with values in E and the finite norm $\| \| u(x) \|_E \|_{L_p(G)}$ [1]. We also employ the Holder spaces (see the definition for instance, in [22]) $C^{\alpha,\beta}(\overline{Q}), C^{\alpha,\beta}(\overline{S})$, $C^k(\overline{G})$ and the Sobolev spaces $W_p^s(G;E)$, $W_p^s(Q;E)$ (see [21, 23]). If $E = \mathbb{R}$ or $E = \mathbb{R}^n$ then the latter space is denoted by $W_p^s(Q)$. Given an interval $J = (0,T)$, put $W_p^{s,r}(Q) = W_p^s(J;L_p(G)) \cap L_p(J;W_p^r(G))$. Respectively, $W_p^{s,r}(S) = W_p^s(J;L_p(\Gamma)) \cap L_p(J;W_p^r(\Gamma))$. All spaces and the coeffciecients of the equation (1) are assumed to be real. Let $(u,v) = \int_G u(x)v(x)dx$, $Q^\gamma = (0,\gamma) \times G$ and $S^\gamma = (0,\gamma) \times \Gamma$.

We endow the space $W_p^s(0,\tau;E)$ ($s \in (0,1)$, E is a banach space, with the norm

$$\| q(t) \|_{W_p^s(0,\tau;E)} = (\| q \|_{L_p(0,\tau;E)}^p + <q>_{s,\tau}^p)^{1/p}, \quad <q>_{s,\tau}^p = \int_0^\tau \int_0^\tau \frac{\| q(t_1) - q(t_2) \|_E^p}{|t_1 - t_2|^{1+sp}} dt_1 dt_2.$$ If $E = \mathbb{R}$ then we

obtain the conventional Sobolev space $W_p^s(0,\tau)$. For $s \in (1/p,1]$, we put $\tilde{W}_p^s(0,\tau) = \{q \in W_p^s(0,\tau): q(0) = 0\}$. This class is a Banach space with the norm $\| \cdot \|_{W_p^s(0,\tau)}$. We can

define also the equivalent norm $\| q(t) \|_{\tilde{W}_p^s(0,\tau)}^p = \| \frac{q}{t^s} \|_{L_p(0,\tau)}^p + <q>_{s,\tau}^p$. The equivalence results from

Lemma 1 of the subsection 3.2.6 [1]. Similarly, we can define the spaces $\tilde{W}_p^s(0,\tau;L_p(G))$, $\tilde{W}_p^{s,2s}(Q^\tau)$ comprising , the functions $v(t,x)$ in $W_p^s(0,\tau;L_p(G))$ and $W_p^{s,2s}(Q^\tau)$, respectively, such that $v(0,x) = 0$. The new norms $\| \cdot \|_{\tilde{W}_p^s(0,\tau;L_p(G))}$, $\| \cdot \|_{\tilde{W}_p^{s,2s}(Q^\tau)}$ are defined naturally with the use of the above norm $\tilde{W}_p^s(0,\tau)$.

Lemma 1. *Let $s \in (1/p,1)$ and $p \in (1,\infty)$. Then the following statements are valid.*

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1) *Let* $q(t) \in W_p^s(0,\tau)$ $(\tau \in (0,T])$. *Then* $q \in C([0,\tau])$ *after a possible change on a set of zero measure. If* $q(0) = 0$ *and* $\tilde{q}$ *is an extension by zero of* q *for* $t \le 0$ *then*

$$\| q \|_{W_p^s(-T+\tau,\tau)} \le c_1 \| q \|_{\tilde{W}_p^s(0,\tau)}, \tag{4}$$

where the constant c_1 *is independent of* $\tau \in (0,T]$ *and* q.

2) *The product* $q \cdot v$ *of functions in* $W_p^s(0,\tau)$ $(\tau \in (0,T])$ *belongs to* $W_p^s(0,\tau)$ *and if* $q \in \tilde{W}_p^s(0,\tau)$ *and* $v \in W_p^s(0,\tau)$ *then* $qv \in \tilde{W}_p^s(0,\tau)$. *Moreover, the following estimate holds:*

$$\| qv \|_{\tilde{W}_p^s(0,\tau)} \le c_2 \| q \|_{\tilde{W}_p^s(0,\tau)} (<v>_{s,\tau} + \| v \|_{L_\infty(0,\tau)}), \tag{5}$$

where the constant c_2 *is independent of* q, v, *and* $\tau \in (0,T]$.

3) *If a function* v *is strictly bounded from zero on* $[0,\tau]$, *i. e.* $\delta_0 = \inf_{t \in [0,\tau]} |v(t)| > 0$ *then the ratio* q/v *of functions in* $W_p^s(0,\tau)$ $(\tau \in (0,T])$ *belongs to* $W_p^s(0,\tau)$ *again and*

$$\| q/v \|_{W_p^s(0,\tau)} \le c_3 \| q \|_{W_p^s(0,\tau)} \| v \|_{W_p^s(0,\tau)}, \tag{6}$$

where the constant c_3 *is independent of* q *but it depends on* δ_0 *and tends to* ∞ *as* $\delta_0 \to 0$.

4) *Assume that* $q(t) \in \tilde{W}_p^s(0,\tau)$ $(\tau \in (0,T])$, $v(t) \in W_p^s(0,T)$, *and* $\Phi(t,x) \in W_p^{s,2s}(S)$. *Then* $qv \in \tilde{W}_p^s(0,\tau)$, $q\Phi \in \tilde{W}_p^{s,2s}(S^\tau)$, *and*

$$\| qv \|_{\tilde{W}_p^s(0,\tau)} \le c \| q \|_{\tilde{W}_p^s(0,\tau)} \| v \|_{W_p^s(0,T)}, \tag{7}$$

$$\| q\Phi \|_{\tilde{W}_p^{s,2s}(S^\tau)} \le c_4 \| q \|_{\tilde{W}_p^s(0,\tau)} \| \Phi \|_{W_p^{s,2s}(S)}, \tag{8}$$

where the constant c is independent of $\tau \in (0,T]$.

The proof can be found in [21].

We describe now the conditions on the data used below. Fix a number $s = 1 - 1/2p$ and assume that $p > 3/2$.

The conditions on the coeficients

$$a_{ij} \in C([0,T]; W_\infty^1(G)) \cap L_q(G; W_p^s(0,T)), \nabla_x a_{ij} \in L_q(G; W_p^s(0,T)), \tag{9}$$

where $i, j = 1, 2, \ldots, n$, $1/p + 1/q = 1$.

$$a_i \in L_q(G; W_p^s(0,T)), (i = 0,1,\ldots,n), \nabla_x a_i \in L_q(G; W_p^s(0,T)) \, (i = 1,2,\ldots,n). \tag{10}$$

Suppose also that there exists a constant $\delta_0 > 0$ such that

$$\sum_{i,j=1}^n a_{ij} \xi_i \xi_j \ge \delta_0 |\xi|^2, \ \forall(t,x) \in Q, \forall \xi \in \mathbb{R}^n. \tag{11}$$

The conditions on the data of the problem

$$f \in L_p(Q), \ u_0(x) \in W_p^{2-2/p}(G), \tag{12}$$

$$g \in W_p^{s,2s}(S), g(0,x) = u_0 |_\Gamma, \tag{13}$$

$$\varphi_k \in W_\infty^2(G), \Phi_k \in W_p^{s,2s}(S), \psi_k \in W_p^{s+1}(0,T), (f, \varphi_k) \in W_p^s(0,T), k = 1,2,\ldots,m. \tag{14}$$

As a consequence of Theorem 9.1 in [22, Ch. 4] (see also Theorem 10.4 in [22, Ch. 7]) we have the following theorem.

Theorem 1. *Assume that* G *is a bounded domain with boundary of the class* C^2 *and the conditions* (9)–(13) *hold. Then there exists a unique solution* u *to the problem* (1)–(2) *such that* $u \in W_p^{1,2}(Q)$. *A solution meets the estimate*

$$\| u \|_{W_p^{1,2}(Q)} \le C(\| f \|_{L_p(Q)} + \| u_0 \|_{W_p^{2-2/p}(G)} + \| g \|_{W_p^{s,2s}(S)}).$$

As a consequece of Theorem 1 we have that

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Theorem 2. *Assume that G be a bounded domain with boundary of the class C^2 and the conditions (9)–(13) hold, where $f \equiv 0$ and $u_0 \equiv 0$. Let $\gamma \in (0,T]$. Then there exists a unique solution u to the problem (1)–(2) such that $u \in W_p^{1,2}(Q^\gamma)$. A solution meets the estimate $\| u \|_{W_p^{1,2}(Q^\gamma)} \leq c \| g \|_{\tilde{W}_p^{s,2s}(S^\gamma)}$, where the constant c is independent of $\gamma \in (0,T]$ and g.*

Proof. Extend the function g by zero for $t < 0$ and put $\tilde{g} = \begin{cases} g(t,x), t \in (-T+\gamma, \gamma) \\ g(2\gamma - t, x), t \in [\gamma, T+\gamma] \end{cases}$. Obviously,

$\tilde{g} \in W_p^{s,2s}(S)$. By Theorem 1, we can construct a solution to the problem (1)–(2), where $u_0 \equiv 0$, $f \equiv 0$, and $g = \tilde{g}$ such that $u \in W_p^{1,2}(Q)$. Theorem 1 yields the estimate $\| u \|_{W_p^{1,2}(Q^\gamma)} \leq c \| \tilde{g} \|_{W_p^{s,2s}(S)}$. Estimate

the right-hand side. Lemma 1 implies that

$\| \tilde{g} \|_{W_p^{s,2s}(S)} \leq \| \tilde{g} \|_{W_p^{s,2s}((-T+\gamma, T+\gamma) \times \Gamma)} \leq c(\| \tilde{g} \|_{W_p^{s,2s}((-T+\gamma, \gamma) \times \Gamma))} + \| \tilde{g} \|_{W_p^{s,2s}((\gamma, T+\gamma) \times \Gamma))}) \leq c_1 \| g \|_{\tilde{W}_p^{s,2s}(S^\gamma)}$.

We employ the additivity of the Sobolev space with respect to the partition of a domain (see Remark 3 of Subsect. 4.4.1 in [1]) and the definition of the corresponding norm.

Basic results

In addition to the above conditions we require that

$$|\det B(t)| \geq \delta_0 > 0 \quad \forall t \in [0,T], \tag{15}$$

where $B(t)$ is the matrix with entries $b_{ij} = \int_\Gamma \frac{\partial \varphi_i(x)}{\partial N} \Phi_j(t,x) d\Gamma$, $\varphi_k |_\Gamma = 0$ ($k = 1,\dots,m$);

$$\int_G u_0(x) \varphi_k(x) dx = \psi_k(0), \psi_{kt}(0) = (u_0, L_0^* \varphi_k) - \int_\Gamma u_0(x) \frac{\partial \varphi_i(x)}{\partial N} d\Gamma + (f, \varphi_k)|_{t=0}, \tag{16}$$

where $k = 1,\dots,m$ and L_0^* is a formally adjoint to the operator L_0,

$L_0 \omega = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} a_{ij} \omega_{x_j} - \sum_{i=1}^n a_i \omega_{x_i} - a_0 \omega$;

(A) the functions $\Phi_1(0,x),\dots,\Phi_m(0,x)$ are linearly independent on Γ and $u_0(x)|_\Gamma$ belongs to the span of these functions.

We can note that (16) is a necessary solvability condition of the inverse problem.

Theorem 3. *Assume that G is a bounded domain with boundary of the class C^2 and the conditions (9)–(12), (14)–(16), and (A) hold. Then there exists a unique solution $(u, \vec{q})$ ($\vec{q} = (q_1,\dots,q_m)$) to the problem (1)–(3) such that $u \in W_p^{1,2}(Q)$, $\vec{q} \in W_p^s(0,T)$. A solution satisfies the estimate*

$$\| u \|_{W_p^{1,2}(Q)} + \| \vec{q} \|_{W_p^s(0,T)} \leq C(\| f \|_{L_p(Q)} + \| u_0 \|_{W_p^{2-2/p}(G)} + \sum_{i=1}^m (\| \psi_i \|_{W_p^{1+s}(0,T)} + \| (f, \varphi_i) \|_{W_p^s(0,T)})).$$

Proof. Let $u \in W_p^{1,2}(Q)$ be a solution to the problem (1)–(3), where $g = \sum_{i=1}^m q_i \Phi_i$. The conditions (15) and (A) imply that there exists a unique collection of constants $q_i(0)$ such that $u_0 |_\Gamma = \sum_{i=1}^m q_i(0) \Phi_i(0,x)$. Put $\sum_{i=1}^m q_i(0) \Phi_i(t,x) = g_0(t,x)$ and denote by $v \in W_p^{1,2}(Q)$ a solution to the problem (see Theorem 1)

$$Lv = f, \ v |_S = g_0(t,x), v |_{t=0} = u_0(x). \tag{17}$$

Let $\vec{q} \in W_p^s(0,T)$. In view of our conditions $\Phi_j \in W_p^{s,2s}(S)$. Lemma 1 yields $q_i(t) \Phi_i(t,x) \in W_p^{s,2s}(S)$ and thus $g \in W_p^{s,2s}(S)$. Make the change of variables $u = v + \omega$. The function $\omega \in W_p^{1,2}(Q)$ is a solution to the problem

$$L\omega = 0, \ B\omega |_S = g - g_0 = \tilde{g}, \omega |_{t=0} = 0. \tag{18}$$

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The condition (3) transforms into the form

$$\int_G \omega \varphi_k(x)dx = \psi_k - \int_G v(t,x)\varphi_k(x)dx = \tilde{\psi}_k, \quad k = 1,2,\ldots,m. \tag{19}$$

By (16), $\tilde{\psi}_k(0) = 0$ and we have at least that $\tilde{\psi}_k(t) \in W_p^1(0,T)$. Below we demonstrate that $\tilde{\psi}_k(t) \in W_p^{1+s}(0,T)$ and $\tilde{\psi}_k'(0) = 0$. Multiply the equation in (18) by $\varphi_k(x)$ and integrate the result over G. We obtain that $(\omega_t, \varphi_k) = (L_0\omega, \varphi_k)$. Using (18), (19), and integrating by parts, we infer

$$\tilde{\psi}_k'(t) = (\omega, L_0^*\varphi_k) - \sum_{i=1}^m \tilde{q}_i(t)\int_\Gamma \Phi_i \frac{\partial \varphi_k(x)}{\partial N}d\Gamma, k = 1,\ldots,m, \tilde{q}_i(t) = q_i(t) - q_i(0).$$

The last inequality can be written in either of the forms

$$\sum_{i=1}^m \tilde{q}_i(t)b_{ki} = -\tilde{\psi}_k'(t) + (\omega, L_0^*\varphi_k), \tag{20}$$

or

$$B\vec{q}_a = \vec{F} + R(\vec{q}_a), \vec{F} = (F_1,\ldots,F_m)^T, F_k = -\tilde{\psi}_k'(t), (R(\vec{q}_a))_k = (\omega, L_0^*\varphi_k), \tag{21}$$

where $\vec{q}_a = (\tilde{q}_1,\ldots,\tilde{q}_m)^T$. The function ω in (21) is a solution to the direct problem (18). The entries of B possess the property $b_{ij} \in W_p^s(0,T)$ and even more we have the inequality

$$\| b_{ij} \|_{W_p^s(0,T)} \leq \| \Phi_j \|_{L_p(\Gamma; W_p^s(0,T))} \| \varphi_i \|_{C^1(\overline{G})}.$$

As was noticed in the proof of Lemma 1, the embedding theorems state that $W_p^s(0,T) \subset C([0,T])$. Hence, we can assume that $b_{ij} \in C([0,T])$. In view of (15), we can write

$$\vec{q}_a = B^{-1}\vec{F} + R_0(\vec{q}_a), \quad R_0(\vec{q}_a) = B^{-1}R(\vec{q}_a). \tag{22}$$

We can determine the vector $\vec{q}_a$ from this equation. Indeed, consider the segment $[0,\delta] \subset [0,T]$ and estimate the quantity $\| R_0(\vec{q}_a) \|_{\tilde{W}_p^s(0,\delta)}$. The second and third statements of Lemma 1 and the conditions on the coeffcients and the functions φ_k imply that the entries of the inverse matrix B^{-1} also belong to the class $W_p^s(0,T)$. In this case the estimate (7) and Lemma 1 yield

$$\| R_0(\vec{q}_a) \|_{\tilde{W}_p^s(0,\delta)} \leq c\sum_{k=1}^m \| (\omega, L_0^*\varphi_k) \|_{\tilde{W}_p^s(0,\delta)}. \tag{23}$$

Estimate the norm of the expression $(\omega, L_0^*\varphi_k)$. The Minkowski and Holder inequalities and Lemma 1 ensure that

$$\| (\omega, L_0^*\varphi_k) \|_{L_p(0,\delta)} \leq \| \omega \|_{L_p(G; \tilde{W}_p^s(0,\delta))} \| L_0^*\varphi_k \|_{L_q(G; W_p^s(0,T))}. \tag{24}$$

In view of our conditions on the coefficients, the last factor is estimated by some constant independent of δ. Estimate the first factor. We have

$$\int_G \| \omega \|_{\tilde{W}_p^s(0,\delta)}^p dx = \int_G\int_0^\delta \frac{|\omega|^p}{t^{sp}}dtdx + \int_G\int_0^\delta\int_0^\delta \frac{|\omega(t_1,x) - \omega(t_2,x)|^p}{|t_1 - t_2|^{1+sp}}dt_1dt_2dx. \tag{25}$$

The Newton–Leibnitz formula validates the inequality

$$\| \frac{1}{t^s}\omega \|_{L_p(Q^\delta)} \leq c_1\delta^{1/2} \| \omega_t \|_{L_p(Q^\delta)}. \tag{26}$$

Estimate the second summand on the right-hand side of (25). To this end, we first make the change of variables $t_1 = \tau_1\delta$, $t_2 = \tau_2\delta$ and next use the inequality $\| \tilde{\omega}(x,\tau) \|_{W_p^s(0,1)} \leq c \| \tilde{\omega}_\tau \|_{L_p(0,1)}$ $(\tilde{\omega}(x,\tau)) = \omega(x,\delta\tau))$ followed by the inverse change of variables. As a result, we arrive at the inequality

$$\int_G\int_0^\delta\int_0^\delta \frac{|\omega(t_1,x) - \omega(t_2,x)|^p}{|t_1 - t_2|^{1+sp}}dt_1dt_2dx \leq c_1\int_G\int_0^\delta |\omega_t(t,x)|dtdx\delta^{1/2}. \tag{27}$$

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Thus, from (25)–(27) it follows that

$$\int_G \|\omega\|_{\tilde{W}_p^s(0,\delta)}^p dx \le c_2\delta^{1/2}\|\omega_t\|_{L_p(Q^\delta)} \le c_2\delta^{1/2}\|\omega\|_{W_p^{1,2}(Q^\delta)}. \tag{28}$$

Therefore, taking (23) and Theorem 2 into account, we obtain the estimate

$$\|R_0(\vec{q}_a)\|_{\tilde{W}_p^s(0,\delta)} \le c_3 c\delta^{1/2}\|\omega\|_{W_p^{1,2}(Q^\delta)} \le c_4\delta^{1/2}\|\tilde{g}\|_{W_p^{s,2s}(Q^\delta)}. \tag{29}$$

Lemma 1 implies the inequality

$$\|R_0(\vec{q}_a)\|_{\tilde{W}_p^s(0,\delta)} \le c_5\delta^{1/2}\|\vec{q}_a\|_{\tilde{W}_p^s(0,\delta)}, \tag{30}$$

where the constant c_5 is independent of δ and $\vec{q}_a$. Fix $\delta > 0$ such that $\delta^{1/2}c_5 < 1$. In this case the operator R_0 is a contraction and thereby the equaiton (22) is uniquely solvable in the space $W_p^s(0,\delta)$, of course under the condition that $\psi_k' \in \tilde{W}_p^s(0,T)$. We have that $\psi_k' \in W_p^s(0,T)$. Show that $\psi_{0k} = \int_G v(t,x)\varphi_k(x)dx \in W_p^{1+s}(0,T)$ and $\psi_{0k}'(0) = \psi_k'(0)$, i. e., $\tilde{\psi}_{kt} \in \tilde{W}_p^s(0,T)$. Multiply the equation in (17) by φ_k and integrate the result over G. We infer

$$\psi_{0k}'(t) = (\omega, L_0^*\varphi_k) - \sum_{i=1}^m q_i(0)\int_\Gamma \Phi_i \frac{\partial\varphi_k(x)}{\partial N}d\Gamma + (f,\varphi_k),, k = 1,\ldots,m. \tag{31}$$

In view of the condition (14) the right-hand side in this equality belongs to $W_p^s(0,T)$ and the relations (16) and (31) for $t = 0$ yield $\psi_{0k}'(0) = \psi_k'(0)$. Thus, $\tilde{\psi}_k \in \tilde{W}_p^s(0,T)$ for all k and thus the equation (22) is uniquely solvable on the segment $[0,\delta]$. Find a solution $\omega \in W_p^{1,2}(Q^\delta)$ to the problem (18). Prove (19). Multiply the equation in (18) by φ_k and integrate the results over G. Using (17), (18), and integrating by parts, we obtain the equality

$$(\omega_t,\varphi_k) = (\omega, L_0^*\varphi_k) - \sum_{i=1}^m \tilde{q}_i(t)\int_\Gamma \Phi_i \frac{\partial\varphi_k(x)}{\partial N}d\Gamma, k = 1,\ldots,m.$$

The vector-function $\vec{q}_a$ satisfies the system (20), subtracting its k-th equation from this equality and cancelling, we arrive at the equality $\int_G \omega_t\varphi_k dx = \tilde{\psi}_k'$, $k = 1,\ldots,m,$ whose integration with respect to t and the intial condition validate (19) on $[0,\delta]$.

We now demonstrate that this solution is extendible to the whole segment $[0,T]$. We have defined the vector-function $\vec{q}_a$ only on $[0,\delta]$. Extend $\vec{q}_a$ by zero for $t < 0$ and denote $\vec{q}_b = \begin{cases} \vec{q}_a(t), t \in (0,\delta) \\ \vec{q}_a(2\delta - t), t \in [\delta,T] \end{cases}$. The coordinates of $\vec{q}_b$ are denoted by $q_1^b,\ldots,q_m^b$. This vector-function belongs to $W_p^{s,2s}(S)$. Make the change $\vec{q}^1 = \vec{q}_a - \vec{q}_b$. The vector-function with the coordinates q_i^1 is a solution to the system

$$\sum_{i=1}^m q_i^1(t)b_{ki} = \tilde{\psi}_k'(t) + (\omega, L_0^*\varphi_k) - \sum_{i=1}^m q_i^b(t)b_{ki}. \tag{32}$$

By definition of $\vec{q}_b$, the right-hand side in this equality and the vector $\vec{q}^1$ vanish on $[0,\delta]$. Let ω_0 – be a solution to the problem

$$L\omega_0 = 0, \ B\omega_0|_S = \sum_{i=1}^m q_i^b\Phi_i, \omega_0|_{t=0} = 0. \tag{33}$$

In this case the function $\omega_1 = \omega - \omega_0$ is a solution to the problem

$$L\omega_1 = 0, \ B\omega_1|_S = \sum_{i=1}^m q_i^1\Phi_i, \omega_1|_{t=0} = 0. \tag{34}$$

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By Theorem 1, $\omega_1 = 0$ for $t \in [0, \delta]$. Thus, the problem of extension of $\vec{q}_a$ is reduced to solving the system

$$\sum_{i=1}^{m} q_i^1(t) b_{ki} = \psi_{1k}'(t) + (\omega_1, L_0^* \varphi_k), \qquad (35)$$

where $\psi_{1k}' = \tilde{\psi}_k'(t) + (\omega_0, L_0^* \varphi_k) - \sum_{i=1}^{m} q_i^b(t) b_{ki}$, and ω_1 is a solution to the problem (34). A solution to the system vanishes for $t \leq \delta$. We obtain the same system with zero Cauchy data at the point $t = \delta$ and a new right-hand side $\vec{F}$. Next, we repeat the same arguments and estimates on the segment $[\delta, 2\delta]$. Without loss of generality, we can assume that the constants arising in estimating the norm of the operator R_0 are the same. Thus, the system (35) is solvable on $[\delta, 2\delta]$. Repeating the arguments on $[2\delta, 3\delta]$ and so on, we can construct a solution on the whole segment $[0, T]$. The estimate in the claim of the theorem has been actually proven in the proof.

Remark. At first sight, the well-posedness conditions (15) look rather strange and possibly arising in the method of the proof. However, employing other methods leads actually to the same condtitions. It is possible that they are essential.

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ОБРАТНЫЕ ЗАДАЧИ ОБ ОПРЕДЕЛЕНИИ ГРАНИЧНЫХ ДАННЫХ С ИНТЕГРАЛЬНЫМИ УСЛОВИЯМИ ПЕРЕОПРЕДЕЛЕНИЯ[1]

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Рассматривается обратная задача об определении вместе с решением начально-краевой задачи для параболического уравнения второго порядка неизвестных функций, входящих в граничное условие Дирихле. Задачи такого вида об определении граничных данных возникают в самых различных задачах математической физики: управление процессами теплообмена и проектирование тепловой защиты, диагностика и идентификация теплопередачи в сверхзвуковых гетерогенных потоках, идентификация и моделирование теплопереноса в теплозащитных материалах и покрытиях, моделирование свойств и тепловых режимов многоразовой тепловой защиты аэрокосмических аппаратов, исследование композиционных материалов и т. п. В качестве условий переопределения берутся интегралы от решения по пространственной области

с весами. Проблема сводится к операторному уравнению типа Вольтерра. Мы устанавливаем теорему существования и единственности решений этой обратной задачи в пространствах Соболева. Решение является регулярным, т. е. все обобщенные производные, входящие в уравнение существуют и суммируемы с некоторой степенью. Доказательство основано на теореме о неподвижной точке и последовательном доказательстве разрешимости на малых промежутках времени. Приведена также оценка устойчивости решений. Полученные условия разрешимости близки к необходимым условиям.

Ключевые слова: обратная задача; параболическое уравнение; краевые и начальные условия; пространство Соболева; теорема существования и единственности; разрешимость.

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