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CALCULATION OF DISCRETE SEMI-BOUNDED OPERATORS' EIGENVALUES WITH LARGE NUMBERS

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In previous works of the article's authors on development of the Galerkin method, linear formulas for calculating the approximate eigenvalues of discrete lower semi-bounded operators have been obtained. The formulas allow calculating the eigenvalues of the specified operators of any number, regardless of whether the eigenvalues of the previous numbers are known or not. At that, it is possible to calculate the eigenvalues with large numbers when application of the Galerkin method is becoming difficult. It is shown that eigenvalues of small numbers of various boundary-value problems, generated by discrete lower semi-bounded operators and calculated by linear formulas and by the Galerkin method, are in a good conformity.

In this paper we use linear formulas to calculate approximate eigenvalues with large numbers of discrete lower semi-bounded operators. Results of calculation of eigenvalues by linear formulas and by known asymptotic formulas for two spectral problems are given. Comparison of the results of calculations of the approximate eigenvalues shows that they almost coincide for sufficiently large numbers. This proves the fact that linear formulas can be used for the considered spectral problems and sufficiently large numbers of eigenvalues.

Keywords: spectral problem; discrete operators; semi-bounded operators; eigenvalues and eigenfunctions of an operator; Galerkin method.

Introduction

It is known that the spectrum of a discrete operator consists of isolated points that have no limit points other than infinity. Moreover, each eigenvalue of a discrete operator has finite multiplicity.

Let L be a discrete semi-bounded from below operator, defined in the separable Hilbert space H. Its eigenvalues μ are determined by finding non-trivial solutions of the equation:

$$Lu = \mu u, \tag{1}$$

which satisfies the given homogeneous boundary conditions. Enumerate them in order of increasing values of eigenvalues, taking into account the multiplicity $\{\mu_n\}_{n=1}^{\infty}$.

To find the eigenvalues of the operator L we use the Galerkin method. Consider a sequence $\{H_n\}_{n=1}^{\infty}$ of finite dimensional spaces $H_n \subseteq H$, which is complete in H. Suppose, that the orthonormal basis of space H_n is known and consists of functions $\{\phi_k\}_{k=1}^{n}$. Wherein the functions ϕ_k must satisfy all boundary conditions of the problem. Following the Galerkin method, we will find the approximate solution of the spectral problem (1) in the form:

$$u_n = \sum_{k=1}^{n} a_k(n)\phi_k. \tag{2}$$

The following theorems were proved in [1].

Theorem 1. *Let L be a discrete semi-bounded from below operator acting in a separable Hilbert space H. If the system of coordinate functions* $\{\phi_k\}_{k=1}^{n}$ *is a basis in the space H, then the Galerkin method applied to the problem of finding the eigenvalues of the spectral problem (1), constructed on this system of functions, converges.*

Theorem 2. *Let L be a discrete semi-bounded from below operator acting in a separable Hilbert space H. If the system of coordinate functions* $\{\phi_k\}_{k=1}^{n}$ *is an orthonormal basis in the space H, then*

$$\tilde{\mu}_n(n) = \left(L\phi_n, \phi_n\right) + \delta_n, \tag{3}$$

where $\delta_n = \sum_{k=1}^{n-1}\left[\tilde{\mu}_k(n-1)-\tilde{\mu}_k(n)\right]$, $\tilde{\mu}_k(n)$ *is the Galerkin approximation of order n to the corresponding eigenvalues* μ_k *of the operator L.*

Formulas (3) allow, as shown in [1], to calculate the approximate eigenvalues of discrete semi-bounded operators with high computational efficiency. Unlike classical methods, they drastically reduce the amount of computation, solve the problem of finding the eigenvalues of any matrices of high order. Also formulas (3) allows to find eigenvalues regardless of whether know eigenvalues with lower numbers or not and solve the problem of calculating all necessary points of the spectrum of discrete semi-bounded operators.

Numerous eigenvalue calculations $\tilde{\mu}_n$ of boundary problems generated by discrete semi-bounded from below operators for $n \le 50$ calculated by formulas (3) and the Galerkin method are in good agreement [1].

In this work, for further verification of the developed methodology for calculating the eigenvalues of discrete semi-bounded operators using formulas (3), we compare the results of their calculation using these formulas with the calculations using known asymptotic formulas for the following spectral problems.

1. Asymptotic formulas for the eigenvalues of the spectral problems under consideration

Consider the classical spectral problem of the form:

$$-y''(x)+q(x)y(x)=\mu y(x), \quad \mu=\lambda^2 \text{ or } \mu=S^2, \quad 0<x<\pi \tag{4}$$

with boundary conditions

$$y(0)=0, \ \ y(\pi)=0, \tag{5}$$

or

$$y(0)=0, \quad y'(\pi)-hy(\pi)=0 \tag{6}$$

with the requirement that the potential $q(x)$, satisfying the condition:

$$\int_0^{\pi} x\left|q(x)\right|dx<\infty.$$

In the thesis of Z.M. Gasimov [2] it was shown, that for eigenvalues μ_n of spectral problems (4), (5) and (4), (6) the following asymptotic formulas:

$$\mu_n=\lambda_n^2, \ \ \lambda_n=n+\frac{1}{\pi n}\int_0^{\pi} q(t)\sin^2(nt)dt+O(r_n^{2)}), \tag{7}$$

$$\mu_n=S_n^2, \ \ S_n=n-0,5-\frac{h}{\pi(n-0,5)}+\frac{1}{\pi(n-0,5)}\int_0^{\pi} q(t)\sin^2\left[(n-0,5)t\right]dt+O(\tilde{r}_n^{2)}) \tag{8}$$

are true respectively. Here:

$$\tilde{r}_n=\frac{1}{n}+r_n, \ \ r_n=\int_0^{2/n} t\left|q(t)\right|dt+\frac{1}{n}\int_{1/2n}^{\pi}\left|q(t)\right|dt.$$

To find the approximate eigenvalues of the spectral problem (4), (5) we construct a system of coordinate functions, each function of which is an eigenfunction of the spectral problem

$$\begin{gathered}-\phi''(x)=\beta\phi(x), \quad 0<x<\pi, \\ \phi(0)=0, \ \ \phi(\pi)=0.\end{gathered} \tag{9}$$

It is not difficult to show, that the spectral problem (9) has a set of eigenvalues $\left\{n^2\right\}_{n=1}^{\infty}$, which corresponds to an orthogonal system of eigenfunctions $\left\{C_n\sin(nx)\right\}_{n=1}^{\infty}$. Constants C_n are found from the normalization conditions.

To find the approximate eigenvalues of the spectral problem (4), (6) we construct a system of coordinate functions, each function of which is an eigenfunction of the spectral problem:

$$-\phi''(x) = \gamma\phi(x), \quad 0 < x < \pi,$$
$$\phi(0) = 0, \quad \phi'(\pi) - hy(\pi) = 0. \qquad (10)$$

The set of eigenvalues $\{\gamma_n\}_{n=1}^{\infty}$ of the spectral problem (10) has no finite limit points. All the eigenvalues are real, non-negative, simple. They are the roots of the transcendental equation

$$\sqrt{\gamma}\cos\left(\pi\sqrt{\gamma}\right) - h\sin\left(\pi\sqrt{\gamma}\right) = 0, \qquad (11)$$

and the corresponding system of eigenfunctions is orthogonal and have the form $\left\{C_n \sin(\sqrt{\gamma_n}x)\right\}_{n=1}^{\infty}$. Constants C_n are found from the normalization conditions.

In case you need to find the eigenvalue γ_n with a sufficiently large number it is difficult to use the transcendental equation (11), because it is necessary to consistently find all the values γ_n with smaller numbers. This leads to a sharp increase in computational calculations. Therefore, in such cases it is necessary to use asymptotic formulas, which can be easily obtained from formulas (8), assuming that $q(t) \equiv 0$:

$$\gamma_n = S_n^2,\ S_n = n - 0,5 - \frac{h}{\pi(n-0,5)} + O\left(\tilde{r}_n^2\right),\ \tilde{r}_n = \frac{1}{n}. \qquad (12)$$

2. Numerical experiments

Denote by $\tilde{\mu}$ the approximate eigenvalues of spectral problems (4), (5) and (4), (6), found by the Galerkin method, by $\hat{\mu}$ the eigenvalues found by formulas (3), by $\bar{\mu}$ the eigenvalues found by asymptotic formulas (7) or (8). In all the above calculations it was assumed that $\delta_n = 0$.

In tables 1 and 2 the eigenvalues of problem (4), (5), found by formulas (3), and asymptotic formulas (7) with potential $q(x) = x^2 - 5x + 13 - \sin(6x) + 2e^x$ are given.

Table 1

n	$\tilde{\mu}_n$	$\hat{\mu}_n$	$\bar{\mu}_n$	$\lvert\tilde{\mu}_n - \bar{\mu}_n\rvert$	$\lvert\hat{\mu}_n - \bar{\mu}_n\rvert$
12	166,712	166,503	167,382	$6{,}705\cdot10^{-1}$	$8{,}792\cdot10^{-1}$
13	191,685	191,507	192,257	$5{,}719\cdot10^{-1}$	$7{,}494\cdot10^{-1}$
14	218,663	218,511	219,157	$4{,}935\cdot10^{-1}$	$6{,}463\cdot10^{-1}$
15	247,646	247,513	248,076	$4{,}302\cdot10^{-1}$	$5{,}632\cdot10^{-1}$
16	278,632	278,515	279,010	$3{,}784\cdot10^{-1}$	$4{,}951\cdot10^{-1}$
17	311,620	311,517	311, 956	$3{,}354\cdot10^{-1}$	$4{,}386\cdot10^{-1}$
18	346,611	346,519	346,910	$2{,}993\cdot10^{-1}$	$3{,}913\cdot10^{-1}$
19	383,602	383,520	383,871	$2{,}687\cdot10^{-1}$	$3{,}512\cdot10^{-1}$
20	422,595	422,521	422,838	$2{,}426\cdot10^{-1}$	$3{,}170\cdot10^{-1}$
…	…	…	…	…	…
43	1871,545	1871,529	1871,598	$5{,}261\cdot10^{-2}$	$6{,}863\cdot10^{-2}$
44	1958,544	1958,529	1958,595	$5{,}025\cdot10^{-2}$	$6{,}554\cdot10^{-2}$
45	2047,544	2047,529	2047,592	$4{,}804\cdot10^{-2}$	$6{,}266\cdot10^{-2}$
46	2138,543	2138,529	2138,589	$4{,}597\cdot10^{-2}$	$5{,}997\cdot10^{-2}$
47	2231,543	2231,529	2231,587	$4{,}404\cdot10^{-2}$	$5{,}744\cdot10^{-2}$
48	2326,542	2326,529	2326,584	$4{,}222\cdot10^{-2}$	$5{,}508\cdot10^{-2}$
49	2423,542	2423,529	2423,582	$4{,}052\cdot10^{-2}$	$5{,}285\cdot10^{-2}$
50	2522,541	2522,529	2522,580	$3{,}892\cdot10^{-2}$	$5{,}076\cdot10^{-2}$
51	2623,541	2623,530	2623,578	$3{,}740\cdot10^{-2}$	$4{,}879\cdot10^{-2}$
…	…	…	…	…	…
63	3991,538	3991,530	3991,562	$2{,}448\cdot10^{-2}$	$3{,}197\cdot10^{-2}$
64	4118,537	4118,530	4118,561	$2{,}366\cdot10^{-2}$	$3{,}098\cdot10^{-2}$
65	4247,537	4247,530	4247,560	$2{,}294\cdot10^{-2}$	$3{,}004\cdot10^{-2}$
66	4378,537	4378,530	4378,559	$2{,}208\cdot10^{-2}$	$2{,}913\cdot10^{-2}$

End of the Table 1

n	$\tilde{\mu}_n$	$\hat{\mu}_n$	$\overline{\mu}_n$	$\lvert\tilde{\mu}_n - \overline{\mu}_n\rvert$	$\lvert\hat{\mu}_n - \overline{\mu}_n\rvert$
67	4511,538	4511,530	4511,558	$2{,}045 \cdot 10^{-2}$	$2{,}827 \cdot 10^{-2}$
68	4646,539	4646,530	4646,558	$1{,}849 \cdot 10^{-2}$	$2{,}744 \cdot 10^{-2}$
69	4783,543	4783,530	4783,557	$1{,}348 \cdot 10^{-2}$	$2{,}665 \cdot 10^{-2}$
70	4922,577	4922,530	4922,556	$2{,}053 \cdot 10^{-2}$	$2{,}590 \cdot 10^{-2}$
71	5063,898	5063,530	5063,555	$3{,}426 \cdot 10^{-1}$	$2{,}517 \cdot 10^{-2}$

Table 2

n	$\hat{\mu}_n$	$\overline{\mu}_n$	$\lvert\hat{\mu}_n - \overline{\mu}_n\rvert$
1000	1000022,531	1000022,531	$1{,}269 \cdot 10^{-4}$
1001	1002023,531	1002023,531	$1{,}267 \cdot 10^{-4}$
1002	1004026,531	1004026,531	$1{,}264 \cdot 10^{-4}$
1003	1006031,531	1006031,531	$1{,}262 \cdot 10^{-4}$
1004	1008038,531	1008038,531	$1{,}259 \cdot 10^{-4}$
…	…	…	…
10000	100000022,531	100000022,531	$1{,}269 \cdot 10^{-6}$
10001	100020023,531	100020023,531	$1{,}267 \cdot 10^{-6}$
10002	100040026,531	100040026,531	$1{,}264 \cdot 10^{-6}$
10003	100060031,531	100060031,531	$1{,}268 \cdot 10^{-6}$
10004	100080038,531	100080038,531	$1{,}268 \cdot 10^{-6}$
…	…	…	…
100000	10000000022,531	10000000022531	$1{,}269 \cdot 10^{-8}$
100001	10000200023,531	10000200023,531	$1{,}269 \cdot 10^{-8}$
100002	10000400026,531	10000400026,531	$1{,}269 \cdot 10^{-8}$
100003	10000600031,531	10000600031,531	$1{,}269 \cdot 10^{-8}$
100004	10000800038,531	10000800038,531	$1{,}269 \cdot 10^{-8}$

Numerical calculations showed that the results of calculations of eigenvalues in three ways are in good agreement. As the number of eigenvalues increases, the difference between them decreases.

The results of calculations for sufficiently large numbers of the eigenvalues of the spectral problem (4), (5) are given in the table 2. The calculation of eigenvalues with such numbers by the Galerkin method causes difficulties due to the large dimensions of the matrices with which you have to work. Therefore, a comparison is made between the approximate eigenvalues found by formulas (3) and the asymptotic formulas (8). For $n > 100\,000$ the values $\hat{\mu}_n$ and $\overline{\mu}_n$ are almost the same.

In Tables 3 and 4 the approximate eigenvalues of the spectral problem (4), (6) calculated by formulas (3) and asymptotic formulas (8) with $h = 0{,}5$ and $q(x) = x^3 - 4x + 5 - \cos(3x) + e^x$ are given.

Table 3

n	$\tilde{\mu}_n$	$\hat{\mu}_n$	$\overline{\mu}_n$	$\lvert\tilde{\mu}_n - \overline{\mu}_n\rvert$	$\lvert\hat{\mu}_n - \overline{\mu}_n\rvert$
8	78,633	78,572	78,742	$1{,}097 \cdot 10^{-1}$	$1{,}740 \cdot 10^{-1}$
9	96,619	96,572	96,708	$8{,}864 \cdot 10^{-2}$	$1{,}362 \cdot 10^{-1}$
10	116,610	116,572	116,683	$7{,}314 \cdot 10^{-2}$	$1{,}114 \cdot 10^{-1}$
11	138,603	138,572	138,665	$6{,}137 \cdot 10^{-2}$	$9{,}280 \cdot 10^{-2}$
12	162,598	162,572	162,650	$5{,}221 \cdot 10^{-2}$	$7{,}849 \cdot 10^{-2}$
…	…	…	…	…	…
36	1338,574	1338,571	1338,581	$6{,}287 \cdot 10^{-3}$	$9{,}177 \cdot 10^{-3}$
37	1412,574	1412,571	1412,580	$5{,}957 \cdot 10^{-3}$	$8{,}694 \cdot 10^{-4}$
38	1488,574	1488,571	1488,580	$5{,}653 \cdot 10^{-3}$	$8{,}248 \cdot 10^{-4}$
39	1566,574	1566,571	1566,579	$5{,}371 \cdot 10^{-3}$	$7{,}836 \cdot 10^{-4}$
40	1646,574	1646,571	1646,579	$5{,}110 \cdot 10^{-3}$	$7{,}454 \cdot 10^{-3}$
…	…	…	…	…	…

End of the Table 3

n	$\tilde{\mu}_n$	$\hat{\mu}_n$	$\overline{\mu}_n$	$\lvert\tilde{\mu}_n - \overline{\mu}_n\rvert$	$\lvert\hat{\mu}_n - \overline{\mu}_n\rvert$
66	4428,572	4428,571	4428,574	$1{,}771\cdot10^{-3}$	$2{,}764\cdot10^{-3}$
67	4562,573	4562,571	4562,574	$1{,}561\cdot10^{-3}$	$2{,}683\cdot10^{-3}$
68	4698,573	4698,571	4698,574	$1{,}104\cdot10^{-3}$	$2{,}605\cdot10^{-3}$
69	4836,577	4836,571	4836,574	$3{,}285\cdot10^{-3}$	$2{,}530\cdot10^{-3}$
70	4976,588	4976,571	4976,574	$1{,}422\cdot10^{-2}$	$2{,}459\cdot10^{-3}$

Table 4

n	$\hat{\mu}_n$	$\overline{\mu}_n$	$\lvert\hat{\mu}_n - \overline{\mu}_n\rvert$
1000	99906,571	999006,571	$1{,}221\cdot10^{-5}$
1001	1001006,571	1001006,571	$1{,}219\cdot10^{-5}$
1002	1003008,571	1003008,571	$1{,}216\cdot10^{-5}$
1003	1005012,571	1005012,571	$1{,}214\cdot10^{-5}$
1004	1007018,571	1007018,571	$1{,}211\cdot10^{-5}$
…	…	…	…
10000	99990006,571	99990006,571	$1{,}220\cdot10^{-6}$
10001	100010006,571	100010006,571	$1{,}220\cdot10^{-6}$
10002	100030008,571	100030008,571	$1{,}219\cdot10^{-6}$
10003	100050012,571	100050012,571	$1{,}219\cdot10^{-6}$
10004	100070018,571	100070018,571	$1{,}219\cdot10^{-6}$
…	…	…	…
100000	9999900006,571	9999900006,571	$1{,}220\cdot10^{-9}$
100001	10000100006,571	10000100006,571	$1{,}220\cdot10^{-9}$
100002	10000300008,571	10000300008,571	$1{,}220\cdot10^{-9}$
100003	10000500012,571	10000500012,571	$1{,}220\cdot10^{-9}$
100004	10000700018,571	10000700018,571	$1{,}220\cdot10^{-9}$

The results of calculations of the approximate eigenvalues of the spectral problem (4), (6), given in Tables 3 and 4 are in good agreement.

Conclusion

Comparison of the results of calculations of the approximate eigenvalues of the spectral problems (4), (5) and (4), (6), carried out according to formulas (3) and asymptotic formulas (7) and (8), show that for sufficiently large numbers the results are almost the same.

In previous papers in the development of the Galerkin method linear formulas for calculating the approximate eigenvalues of discrete semibounded from below operators were obtained by the authors of the article. Formulas allow you to calculate the eigenvalues of the specified operators with any of their numbers, regardless of whether the eigenvalues with the previous numbers are known or not. In this case, it is possible to calculate the eigenvalues with large numbers, when the application of the Galerkin method becomes difficult. To test the new method for calculating the eigenvalues of discrete semi-bounded operators, computational experiments were conducted, which showed that the eigenvalues of small numbers of various boundary-value problems calculated by linear formulas and the Galerkin method are in good agreement. For further verification of the obtained linear formulas, it became necessary to find out how they behave when calculating eigenvalues with large numbers when asymptotic formulas begin to work. In this paper we use linear formulas to calculate approximate eigenvalues with large numbers of discrete semi-bounded from below operators. The results of calculating the eigenvalues by linear formulas and by known asymptotic formulas for two spectral problems are given. Comparison of the results of the calculations of the approximate eigenvalues show that for sufficiently large numbers they almost coincide. This confirms the fact that linear formulas can be used for the considered spectral problems and sufficiently large numbers of eigenvalues.

In the spectral problems considered, linear formulas give the same result as asymptotic formulas. This confirms the possibility of applying linear formulas to the approximate calculation of any eigenvalue of a discrete semi-bounded operator. By virtue of the linearity of formulas, finding eigenvalues becomes computationally efficient compared to any classical method.

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ВЫЧИСЛЕНИЕ СОБСТВЕННЫХ ЗНАЧЕНИЙ С БОЛЬШИМИ НОМЕРАМИ ДИСКРЕТНЫХ ПОЛУОГРАНИЧЕННЫХ ОПЕРАТОРОВ

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В предыдущих работах авторов статьи в развитии метода Галеркина получены линейные формулы для вычислений приближенных собственных значений дискретных полуограниченных снизу операторов. Формулы позволяют вычислять собственные значения указанных операторов любого номера независимо от того, известны ли собственные значения с предшествующими номерами или нет. При этом можно вычислять собственные значения и с большими номерами, когда применение метода Галеркина становится затруднительным. Показано, что собственные значения небольших номеров различных краевых задач, порожденных дискретными полуограниченными снизу операторами, вычисленные по линейным формулам и методом Галеркина, хорошо согласуются.

В работе применены линейные формулы для вычисления приближенных собственных значений с большими номерами дискретных полуограниченных снизу операторов. Приведены результаты вычислений собственных значений по линейным формулам и по известным асимптотическим формулам для двух спектральных задач. Сравнение результатов проведенных вычислений приближенных собственных значений показывает, что для достаточно больших номеров они практически совпадают. Это подтверждает тот факт, что для рассматриваемых спектральных задач и достаточно больших номеров собственных значений можно использовать линейные формулы.

Ключевые слова: спектральная задача; дискретные операторы; полуограниченные операторы; собственные числа и собственные функции оператора; метод Галеркина.

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