Математика

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CLASSIFICATION OF PRIME PROJECTIONS OF KNOTS IN THE THICKENED TORUS OF GENUS 2 WITH AT MOST 4 CROSSINGS

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We begin classification of prime knots in the thickened torus of genus 2 having diagrams with at most 4 crossings. To this end, it is enough to construct a table of prime knot projections with at most 4 crossings, and use the table to obtain table of prime diagrams, i. e. table of prime knots. In this paper, we present the result of the first step, i. e. we construct a table of prime projections of knots in the thickened torus of genus 2 having at most 4 crossings. First, we introduce definition of prime projection of a knot in the thickened torus of genus 2. Second, we construct a table of prime projections of knots in the thickened torus of genus 2 having at most 4 crossings. To this end, we enumerate graphs of special type and consider all possible embeddings of the graphs into the torus of genus 2 that lead to prime projections. In order to simplify enumeration of the embeddings, we prove some auxiliary statements. Finally, we prove that all obtained projections are inequivalent. Several known and new tricks allow us to keep the process within reasonable limits and rigorously theoretically prove the completeness of the constructed table.

Keywords: prime projection; knot; thickened torus of genus 2; table.

Introduction

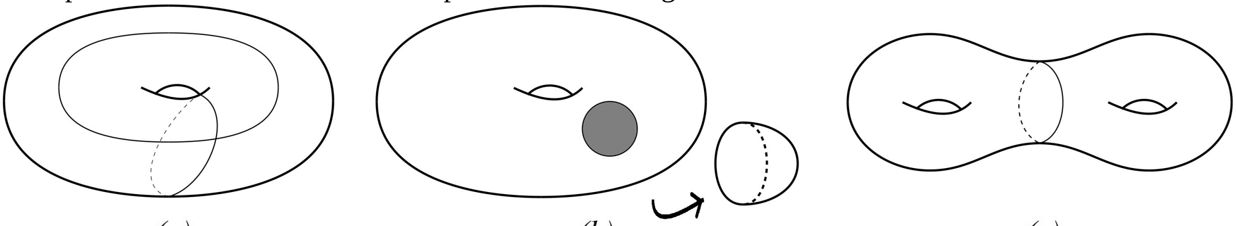
One of the main problems of the knot theory is to find an algorithm to recognize a knot (or link), i. e., to provide the studied object with a unique identifier. For instance, the identifier can be given by a catalog number. This approach involves the problem to construct complete tables of knots and links arranged with respect to some their numerical characteristics. Many researchers worked in this aria during last 150 years. Most of the constructed tables consider knots and links in the 3-dimensional sphere, see [1–3]. Recently, increasing interest in the theory of global knots (i. e., knots in arbitrary 3-manifolds) leads to tabulation of knots in manifolds different from the 3-dimensional sphere. Note tables of links in the projective space [4], knots in the solid torus [5], knots in the thickened Klein bottle [6], as well as prime knots in the lens spaces [7]. Note that, in the knot theory, recent tables includes only the so-called prime objects, which can not be obtained by some known operations from already tabulated objects. Virtual knots and knots in the thickened surfaces have been of particular interest in the last 20 years. Therefore, some tables of such knots were also constructed. In particular, the works [8] and [9] present perfect tables of virtual knots arranged with respect to number of classical crossing and construct a list of some properties of each knot. However, these tables are constructed without taking into account primeness and such important property of a knot as the genus determined by the minimal genus of the thickened surface which can contain the given knot. The natural idea is to classify virtual knots taking into account both parameters, i. e. not only number of classical crossings, but also the genus of a knot, see the papers [10, 11] for tables of prime knots and links in the thickened torus. In a sense, such tables can be considered as tables of prime virtual knots and links of genus 1.

We begin classification of prime knots in the thickened torus of genus 2. To this end, in this paper, we present the result of the first step, i. e. we construct a table of prime projections of knots in the thickened torus of genus 2 having at most 4 crossings. Our main result states that there exist exactly 14 pairwise inequivalent such projections. Further, we intend to use the obtained table of prime projections in order to construct table of prime diagrams, i. e. table of prime knots.

The paper is organized as follows. Section 1 gives required definitions and the main result of the paper. Section 2 describes main ideas of the tabulation of prime projections of knots in the thickened torus of genus 2.

1. Main Result

A direct product of two copies of an 1-dimensional sphere S^1 is called a 2-*dimensional torus* $T = S^1 \times S^1$. Further, for shortness, we refer to a 2-dimensional torus T as a torus T. Fig. 1, *a* shows an example of a torus T endowed with a pair "meridian-longitude" of T.



(a) (b) (c)

Fig. 1. (a) A torus *T* endowed with a pair "meridian-longitude", *(b)* a torus T^o with a hole and a disk D, *(c)* a *2-dimensional torus* T_2 *of genus 2* formed by a connected sum of two copies of a torus T^o with a hole

A *surface* F^o *with a hole* is obtained from the original surface F by removing the interior of a 2-dimensional disk D. Further, for shortness, we refer to a 2-dimensional disk D as a disk D. Fig. 1, *b* shows an example: a torus T^o with a hole is obtained from a torus T by removing the interior of a disk D. Hereinafter, we write ^{o} to show that a surface has one hole, ^{oo} to show that a surface has two holes, etc.

By a 2-*dimensional torus* T_2 *of genus 2* we mean a surface formed by a sum of two copies of a 2-dimensional torus T^o with a hole constructed by identifying (gluing together) their holes, see Fig. 1, *c*. Here each torus T^o is called a handle of a 2-dimensional torus T_2 of genus 2. Further, for shortness, we refer to a 2-dimensional torus T_2 of genus 2 as a torus T_2.

Let us define types of simple closed circles, which can be considered in a torus T_2.

A simple closed circle $C \subset T_2$ is said to be *cut*, if the complement $T_2 \backslash C$ consists of two components.

In the torus T_2, a cut circle C can be either *trivial*, i. e. bounding a disk D, or *nontrivial*. In the first case, the complement $T_2 \backslash C$ is formed by a disk D and a torus $T_2{}^o$ with a hole. In the second case, the complement $T_2 \backslash C$ is formed by two copies of a torus T^o with a hole.

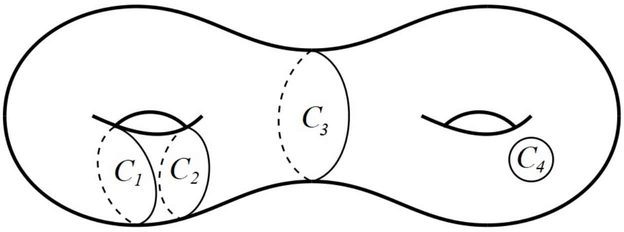


Fig. 2. Examples of circles in the *torus* T_2

A simple closed circle $C \subset T_2$ is said to be *noncut*, if the complement $T_2 \backslash C$ consists of the unique component. Namely, the complement $T_2 \backslash C$ is a torus T^{oo} with two holes.

Two noncut simple closed circles C_1, $C_2 \subset T_2$ are said to be *parallel to each other*, if the complement $T_2 \backslash (C_1 \cup C_2)$ consists of two components, which are a torus T^{oo} with two holes and an *annulus* A, i. e. a 2-dimensional sphere S^{oo} with two holes.

Fig. 2 shows examples: $C_1, C_2 \subset T_2$ are two noncut circles parallel to each other, while C_3, $C_4 \subset T_2$ are nontrivial and trivial cut circles, respectively.

Consider a torus T_2 and an interval $I = [0, 1]$. By a *thickened torus of genus 2* we mean a 3-dimensional manifold homeomorphic to the direct product $T_2 \times I$.

A smooth embedding of the set of m pairwise disjoint circles in the interior $Int(T_2 \times I)$ of the thickened torus $T_2 \times I$ is called an *m-component link* in $T_2 \times I$. In particular, if $m=1$, we have a smooth embedding of the unique circle in $Int(T_2 \times I)$, which is called a knot in $T_2 \times I$ and denoted by $K \subset T_2 \times I$.

As in the classical case, knots in the thickened torus $T_2 \times I$ can be given by their *diagrams*, which are defined by analogy with a classical knot diagram except that a knot is projected into the torus T_2 instead of a 2-dimensional sphere S^2.

A *projection* of a knot K in the torus T_2 is a diagram of K such that the crossings of the diagram contain no under/over-crossing information. Therefore, a projection can be considered as an embedding of a connected regular graph of degree 4, i. e. valence of each vertex of the graph is equal to 4. Vertices of G are called *crossings* of G, while connected components of the complement $T_2 \backslash G$ are called *faces* of G.

Two projections G and G' in the torus T_2 are said to be *equivalent*, if there exists a homeomorphism $f: T_2 \to T_2$ such that $f(G) = G'$.

We say that an intersection point P of two circles $C_1, C_2 \subset T_2$ is *nontransversal*, if only two of four angles near P are formed by both circles C_1 and C_2, while the third and the forth angles are formed only by the circle C_1 and C_2 , respectively. Otherwise, i. e. if all four angles near P are formed by both circles C_1 and C_2, the intersection point P is called *transversal*.

We define the following three types of projections in the torus T_2.

1. The projection G is called *essential*, if each face of G is homeomorphic to a disk D.
2. The projection G is called *composite*, if at least one of the following conditions holds.

(a) There exists a disk $D \subset T_2$ such that the boundary ∂D intersects G transversally exactly in two points, which are internal for two distinct edges of G, and at least one vertex of G is inside D.

(b) There exist two parallel noncut simple closed circles C_1, $C_2 \subset T_2$, and two distinct edges e_1, e_2 of G such that for i = 1, 2 the circle C_i intersects the edge e_i transversally at exactly one internal point, and both surfaces (a torus T^{oo} with two holes and an annulus A) to which the circles divide the torus T_2 contain vertices of G.

(c) There exists nontrivial cut simple closed circle C and two distinct edges e_1 ,e_2 of G such that for i = 1, 2 the circle C intersects the edge e_i transversally at exactly one internal point, and both surfaces (two copies of a torus T^o with a hole) to which the circle C divide the torus T_2 contain vertices of G.

3. The projection G is called *prime*, if G is essential and noncomposite.

Our table contains only prime projections. Indeed, nonessential projections correspond to knots that can be found in already existing tables of knots in the 3-dimensional sphere S^3 [1–3], thickened annulus $A\times I$ (solid torus) [5], or thickened torus $T\times I$ [10]. In its turn, composite projections correspond to knots, which can be constructed using already known knots mentioned above. Namely, composite projections of types (a)–(c) correspond to knots, which can be constructed as sums of a classical knot and a knot in the thickened torus $T_2\times I$, a knot in the thickened torus $T_2\times I$ and a knot in the thickened torus $T\times I$, or two knots in the thickened torus $T\times I$, respectively.

Theorem 1. In the torus T_2, there exist exactly 14 pairwise inequivalent prime projections with at most 4 crossings. The projections are given in Fig. 5.

Theorem 1 is proved by three steps described in Section 2.

2. Proof of the main result

Let us describe main ideas of the tabulation of prime projections given in this section. We do this in three steps. First, Subsection 2.1 enumerates graphs of special type. Then, Subsection 2.2 considers all possible embeddings of the graphs into the torus T_2 giving prime projections. Finally, Subsection 2.3 proves that all constructed projections are pairwise inequivalent.

2.1. Enumeration of graphs with at most 4 vertices whose embeddings into the torus T_2 can be prime projections

Lemma 1. If a projection $G \subset T_2$ is prime, then G is connected and contains no loop nor any cut pair of edges (i. e., removing the pair of edges gives a disconnected graph).

Proof of Lemma 1 is similar to arguments used to prove Lemma 2 in [11].

Lemma 2. Let $G \subset T_2$ be a prime projection with n crossings, then G contains exactly (n–2) faces.

Proof. Take into account the Euler characteristic of the torus T_2 and the fact that G is essential.

Lemma 3. There exist exactly 3 graphs with at most 4 vertices whose embeddings into the torus T_2 can be prime projections, see graphs a – c given in Fig. 3.

Proof. By virtue of Lemma 2, it is easy to see that any graph which embedding into the torus T_2 gives a prime projection contains at least 3 vertices. Lemma 1 gives conditions on an abstract quadrivalent graph, which embedding into the torus T_2 gives a prime projection. All graphs with at most 4 vertices satisfying the first and second conditions are enumerated in [10]. In this list, there are exactly 3 graphs satisfying the third condition, see graphs a−c given in Fig. 3.

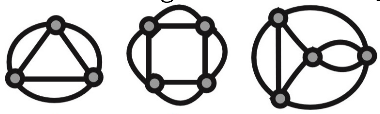


Fig. 3. The graphs of special type

2.2. Construction of prime projections

Lemma 4. All projections shown in Fig. 5 can be obtained as embeddings of the graphs a – c given in Fig. 3. Namely, the graph a gives the projection 3_1, the graph b gives the projections 4_1 and 4_6, and the graph c gives the projections 4_2 -4_5 and 4_7 -4_{13}.

Proof. We construct all the projections by the following method [11].

Let $G \subset T_2$ be a prime projection represented as a union U of the circles C_i, i = 1, 2,…, m, having k nontransversal points.

Let l_1, l_2 be small arcs containing a nontransversal point of the projection G. We can remove the point by the move M shown in Fig. 4. The dashed arc β shows how to perform the inverse move M^{-1}.

Remove each nontransversal point of the projection G by the move M. The obtained union U^k of the same circles C_i, i = 1, 2,…, m contains only transversal points and is endowed with k dashed arcs β to show where the move M was performed. Of course, the initial projection G can be obtained from U^k by the inverse move M^{-1} performed along each dashed arc β, see Fig. 4.

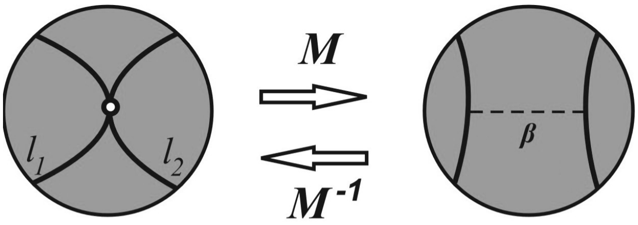


Fig. 4. Move *M* removes a nontransversal point, while *M*⁻¹ is performed along the dashed arc β and creates the point

According to Lemma 3, all prime projections in the torus T_2 with at most 4 crossings can be obtained as embeddings of the graphs *a*–*c*. In order to construct all the projections, we represent an embedding of each graph as a union of a number of circles and enumerate all possible combinations of types of circles and intersection points.

Let us give three obvious statements, which allow to reduce such enumeration.

Lemma 5. Let $G \subset T_2$ be a prime projection represented as a union of circles with n intersection points. Then

(i) this union contains no more than (n–3) cut circles,

(ii) for $n \leq 4$, all cut circles are trivial.

Proof. Statement (i) is true according to Lemma 2 and the fact that each cut circle involves an additional face. If Statement (ii) is not satisfied, then G is either a link projection, or a nonessential projection. This completes the proof of Lemma 5.

Lemma 6. Let $G \subset T_2$ be a prime projection represented as a union U of circles, and $C \subset U$ be a circle having exactly two intersection points with other circles embedded in the torus T_2. Then both points are nontransversal, if C is cut, and at least one of two points is nontransversal, if C is noncut.

Proof. If both intersection points are transversal, then C forms a projection of a component of a link, while we consider only projections of knots. If C is cut, then both intersection points are either transversal, or nontransversal. This completes the proof of Lemma 6.

Lemma 7. Let $G \subset \mathrm{T}_2$ be a prime knot projection obtained from the union U^k, which is endowed with k dashed arcs β. Then the following conditions hold.

(i) The union of U^k and all k dashed arcs β divide the torus T_2 into disks.

(ii) For any two circles C_1, $C_2 \subset U^k$, there exists a sequence of dashed arcs β and other circles that connect C_1 and C_2.

Proof. If Condition (i) is not satisfied, then the projection G is nonessential, and we arrive at contradiction with the fact that the projection G is prime. If Condition (ii) is not satisfied, then G is a projection of a link, while G is a knot projection. This completes the proof of Lemma 7.

Let us enumerate all possible embeddings of the graphs *a*–*c* giving prime projections.

Graph a. Let the projection G be an embedding of the graph a in the torus T_2. The pairs of double edges form three circles in the torus T_2 such that each circle has exactly one intersection point with each of two other circles. According to Lemma 5, all circles are noncut. Note that there exists at most 1 transversal intersection point, otherwise there exists a circle with two transversal intersection points and we arrive at contradiction with Lemma 6.

Case 1. If all intersection points are nontransversal, then we remove all the points by the move M and see that the three circles without common points divide the torus T_2 into more than one part, i. e. we arrive at contradiction with Lemma 2.

Case 2. If exactly one of three intersection points is transversal, then, without loss of generality, we consider the circles C_1 and C_2 to be a pair "meridian-longitude" of one of the handles of the torus T_2, and the circle C_3 to be a meridian of another handle. We remove both nontransversal points by the move M and cut the torus T_2 along all the three circles to obtain a sphere S^{ooo} with three holes. By virtue of Lemma 7, there exists the unique way to draw two dashed arcs β such that to connect each of two holes corresponded to the circle C_3 with the hole formed by the circles C_1 and C_2 under the condition that there exists exactly one endpoint of a dashed arc β on each circle C_i, i = 1, 2. Apply the inverse move M^{-1} along each dashed arc β and obtain the projection $\mathbf{3_1}$.

Graph b. Let the projection G be an embedding of the graph b in the torus T_2. The pairs of double edges form four circles in the torus T_2 such that each circle has exactly one intersection point with each of the two other circles and does not intersect the fourth circle. Note that there exist at most 2 transversal intersection points (moreover, each circle contains no more than one transversal intersection point), otherwise there exists a circle with two transversal intersection points and we arrive at contradiction with Lemma 6.

Case 1. Suppose that all four intersection points are nontransversal.

Remove all the points by the move M and see that the four circles without common points divide the torus T_2 into more than two parts, i. e. we arrive at contradiction with Lemma 2.

Case 2. Suppose that exactly one of four intersection points is transversal.

According to Lemma 5, there exists at most one cut circle, moreover, this circle is trivial. Therefore, without loss of generality, we consider the circles C_1 and C_2 to be a pair "meridian-longitude" of one of the handles of the torus T_2, and the circle C_3 to be a meridian of another handle, while the circle C_4 can be either cut or noncut. We remove all three nontransversal points by the move M and cut the torus T_2 along all the four circles.

Case 2.1. If the circle C_4 is cut, then we obtain a sphere S^{oooo} with four holes. By virtue of Lemma 7, there exists the unique way to draw three dashed arcs β such that to connect one of two holes corresponded to the circle C_3 and the hole corresponded to the circle C_4 with the hole formed by the circles C_1 and C_2 and to connect another hole corresponded to the circle C_3 with the hole corresponded to the circle C_4 under the condition that there exists exactly one endpoint of a dashed arc β on each circle C_i, $i = 1, 2$. Apply the inverse move M^{-1} along each dashed arc β and obtain the projection $\mathbf{4_1}$.

Case 2.2. If the circle C_4 is noncut, then we obtain a sphere S^{ooo} with three holes and an annulus A. In S^{ooo}, by virtue of Lemma 7, there exists the unique way to draw two dashed arcs β such that to connect the hole corresponded to the circle C_3 and the hole corresponded to the circle C_4 with the hole formed by the circles C_1 and C_2 under the condition that there exists exactly one endpoint of a dashed arc β on each circle C_i, $i = 1, 2$. In A, by virtue of Lemma 7, there exists the unique way to draw a dashed arc β such that to connect the hole corresponded to the circle C_3 and the hole corresponded to the circle C_4. Apply the inverse move M^{-1} along each dashed arc β and obtain the projection $\mathbf{4_6}$.

Case 3. Suppose that exactly two of four intersection points are transversal.

According to Lemma 6, these transversal points belong to different pairs of circles. Therefore, without loss of generality, we consider the circles C_1 and C_2 to be a pair "meridian-longitude" of one of the handles of the torus T_2, while the circles C_3 and C_4 form a pair "meridian-longitude" of another handle. We remove both nontransversal points by the move M and cut the torus T_2 along all the four circles to obtain an annulus A. By virtue of Lemma 7, there exists no ways to draw two dashed arcs β such that to connect two holes under the condition that there exist exactly one endpoint of a dashed arc β on each circle C_i, $i = 1, 2, 3, 4$.

Graph c. Let the projection G be an embedding of the graph c in the torus T_2, then G can be represented as a union of three circles such that the circles C_1 and C_2 have no common points, while the circle C_3 intersects each of them alternately. Further, without loss of generality, we consider the circle C_1 to be a representative of the circles C_1 and C_2. Note that there exist at most 2 transversal intersection points (moreover, each of the circles C_1 and C_2 contains no more than one transversal intersection point), otherwise there exists a circle with two transversal intersection points and we arrive at contradiction with Lemma 6. Also, according to Lemma 5, there exists at most one cut circle, moreover, this circle is trivial.

Case 1. Suppose that all intersection points are nontransversal.

Case 1.1. Suppose that there exists no cut circles and consider all possible cases of parallel circles.

If there are no parallel circles, then we remove all the nontransversal points by the move M and cut the torus T_2 along all the three circles to obtain two copies of a sphere S^{ooo} with three holes. In each S^{ooo}, by virtue of Lemma 7, there exists the unique way to draw two dashed arcs β such that to connect the hole corresponded to the circle C_3 with each of the other holes. Apply the inverse move M^{-1} along each dashed arc β and obtain the projection $\mathbf{4_{12}}$.

If the circles C_1 and C_2 are parallel to each other, then they bound an annulus A without crossings, therefore, the projection G is nonessential, and, therefore, G is nonprime.

If the circles C_1 and C_3 are parallel to each other, then we remove all the nontransversal points by the move M and cut the torus T_2 along all the three circles to obtain a sphere S^{ooo} with three holes and an annulus A. In S^{ooo}, by virtue of Lemma 7, there exists the unique way to draw three dashed arcs β such that to connect the hole corresponded to the circle C_3 with each of the other holes. In A, by virtue of Lemma 7, there exists the unique way to draw a dashed arc β such that to connect two holes. Apply the inverse move M^{-1} along each dashed arc β and obtain the projection $\mathbf{4_{10}}$.

Case 1.2. Suppose that there exists a cut circle and remove all the points by the move M to obtain a sphere S^{ooooo} with five holes.

If the cut circle is C_1, then, by virtue of Lemma 7, there exists the unique way to draw four dashed arcs β such that to connect each of two holes corresponded to the circle C_2 with the corresponded hole formed by the circle C_3 and to connect the hole formed by the circle C_1 with both holes formed by the circle C_3. Apply the inverse move M^{-1} along each dashed arc β and obtain the projection $\mathbf{4_3}$.

If the cut circle is C_3, then, by virtue of Lemma 7, there exists the unique way to draw four dashed arcs β such that to connect each of holes corresponded to the circles C_1 and C_2 with the hole formed by the circle C_3 alternately. Apply the inverse move M^{-1} along each dashed arc β and obtain a projection of a link.

Case 2. Suppose that exactly one of four intersection points is transversal. Without loss of generality, we consider the circles C_1 and C_3 to be a pair "meridian-longitude" of one of the handles of the torus T_2, while the circle C_2 can be either cut or noncut. We remove all three nontransversal points by the move M and cut the torus T_2 along all the three circles.

If the circle C_2 is cut, then we obtain a torus T^{oo} with two holes. By virtue of Lemma 7, there exists the unique way to draw three dashed arcs β such that to connect twice the hole corresponded to the circle C_2 with the hole formed by the circles C_1 and C_3, and the last hole with itself. Apply the inverse move M^{-1} along each dashed arc β and obtain the projection $\mathbf{4_2}$.

If the circle C_2 is noncut, then, without loss of generality, we consider the circle C_4 to be a meridian of another handle of the torus T_2, and obtain a sphere S^{ooo} with three holes. By virtue of Lemma 7, there exist two ways to draw three dashed arcs β such that to connect each of the holes corresponded to the circle C_2 and with the hole formed by the circles C_1 and C_3, and the last hole with itself alternately. Indeed, the third possible way leads to a link projection. Apply the inverse move M^{-1} along each dashed arc β and obtain the projections $\mathbf{4_5}$ and $\mathbf{4_8}$.

Case 3. Suppose that exactly two of four intersection points are transversal.

According to Lemma 6, each of the circles C_1 and C_2 contains exactly one transversal point, therefore, both the circles C_1 and C_2 are noncut. The circle C_3 is also noncut, since the circles C_1 and C_2 do not intersect each other and the circle C_3 has exactly one transversal point with the circle C_i , where $i = 1, 2$.

Case 3.1. Suppose that the circles C_1 and C_2 are parallel to each other. Without loss of generality, we consider the circles C_1 and C_2 to be two meridians of one of the handles of the torus T_2, while the circle C_3 is a longitude of the same handle. We remove both nontransversal points by the move M and cut the torus T_2 along all three circles to obtain a torus T^o with a hole and a disk D. In D, there exists no dashed arcs β, otherwise we obtain either a link projection, or a nonessential (therefore, nonprime) knot projection. In T^o, by virtue of Lemma 7, there exists the unique way to connect the hole with itself by two dashed arcs β such that there exist exactly one endpoint of a dashed arc β on each circle C_i, $i = 1, 2$, and two endpoints of different dashed arcs β on the circle C_3. Apply the inverse move M^{-1} along each dashed arc β and obtain the projection $\mathbf{4_9}$.

Case 3.2. Suppose that the circles C_1 and C_2 are not parallel to each other. Without loss of generality, we consider the circle C_i to be a meridian of the i-th handle of the torus T_2, where i =1, 2, while the circle C_3 is a connected sum of two longitudes of both handles. We remove both nontransversal points by the move M and cut the torus T_2 along all three circles to obtain a sphere S^{oo} with two holes. By virtue of Lemma 7, there exist exactly four possible ways to connect the two holes by two dashed arcs β such that there exist exactly one endpoint of a dashed arc β on each circle C_i, $i = 1, 2$, and two endpoints of different dashed arcs β on the circle C_3. These four ways are different in the sense of the following two facts. First, either there exists a dashed arc β connecting a hole with itself, or both dashed arcs β connect different holes. Second, either there exists a fragment of the circle C_3 having both endpoints of different dashed arcs β, or there exist two fragments of the circle C_3 that belong to the same hole, and each fragment contains an endpoint of a dashed arc β. Apply the inverse move M^{-1} along each dashed arc β and obtain the projections $\mathbf{4_4}$, $\mathbf{4_7}$, $\mathbf{4_{11}}$, and $\mathbf{4_{13}}$. This completes the proof of Lemma 4.

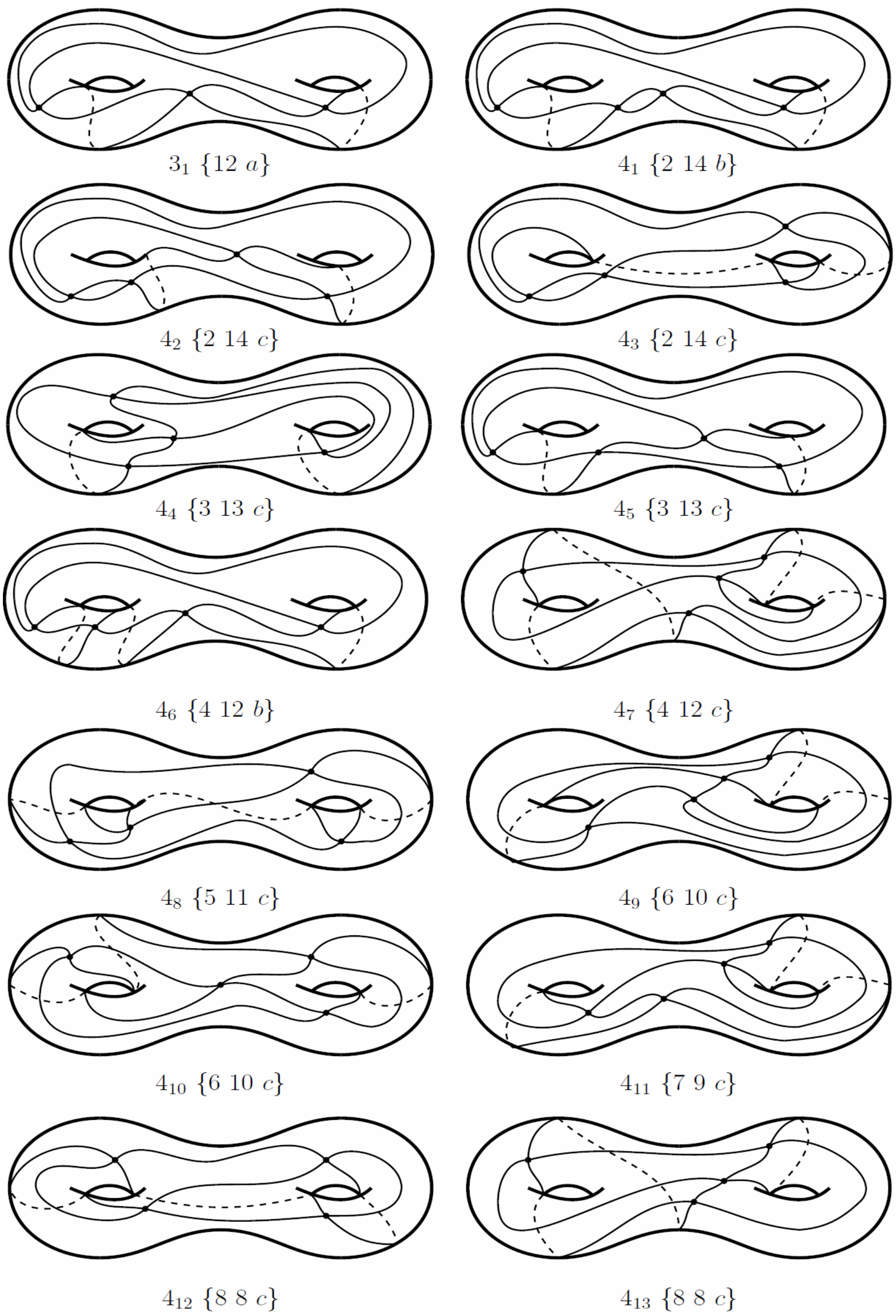


Fig. 5. Prime knot projections in the torus T_2 with at most 4 crossings

3.3. Proof of the fact that all constructed projections are pairwise inequivalent

Lemma 8. All 14 projections given in Fig. 5 are pairwise inequivalent.

Proof. We associate each face of a projection with a natural number, which is equal to the number of edges which form the boundary of the face. Each face of a prime projection is homeomorphic to a disk. According to Lemma 2, the number of faces of each projection given in Fig. 5 is equal to 2 with the exclusion of the projection $\mathbf{3_1}$.

Associate each projection (except for the projection $\mathbf{3_1}$) given in Fig. 5 with an ordered set $\{i_1\, i_2\, x\}$, where i_1 and i_2 are natural numbers, which are associated with faces of the projection and taking in nondecreasing order, and x is the graph such that the projection is an embedding of x in the torus T_2, $x \subset \{b, c\}$. By analogy, the projection $\mathbf{3_1}$ is associated with the ordered set $\{12\ a\}$.

Such ordered sets are enough to prove that all projections given in Fig. 5 are pairwise inequivalent, with the exception of the following 4 pairs: $(4_2, 4_3)$, $(4_4, 4_5)$, $(4_9, 4_{10})$, and $(4_{12}, 4_{13})$.

Let us prove that projections in each of the pairs are also inequivalent. We say that an edge e of the projection G has type (i, j) if e is a common edge of the i-gonal and j-gonal faces of the projection G.

1. Projections $(4_2, 4_3)$ are inequivalent. Indeed, recall that the "straight ahead" rule determines a cycle composed of all edges of the projection. Only in 4_3, the cycle is such that there are the same number of edges of type (14, 14) between two edges of type (2, 14).

2. Projections $(4_4, 4_5)$ are inequivalent, because there exists no bijective mapping between their Gauss codes: 12324143 and 12324134, respectively.

3. Projections $(4_9, 4_{10})$ are inequivalent, because only 4_9 contains the edge of type (6, 6), while the type of each edge of 4_{10} is either (10, 10), or (6, 10).

4. Projections $(4_{12}, 4_{13})$ are inequivalent, because only 4_{13} contains the edges that are common for the same 8-gonal face, while all edges of 4_{12} are common for both 8-gonal faces.

Note that all tabulated projections are prime by construction.

This completes the proof of both Lemma 8 and Theorem 1.

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КЛАССИФИКАЦИЯ ПРИМАРНЫХ ПРОЕКЦИЙ УЗЛОВ В УТОЛЩЕННОМ ТОРЕ РОДА 2 С НЕ БОЛЕЕ ЧЕМ 4 ПЕРЕКРЕСТКАМИ[1]

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Мы начинаем классификацию примарных узлов в утолщенном торе рода 2, имеющих диаграммы с не более чем 4 перекрестками. Классификация проводится в два шага. На первом шаге строится таблица примарных проекций с не более чем 4 перекрестками. На втором шаге полученная таблица используется для построения таблицы примарных диаграмм, т.е. таблицы примарных узлов. В этой статье мы представляем результат первого шага, т.е. строим таблицу всех примарных проекций узлов в утолщенном торе рода 2, имеющих не более 4 перекрестков. Таблица строится в три этапа. На первом этапе мы вводим определение примарной проекции узла в утолщенном торе рода 2. На втором этапе мы строим таблицу примарных проекций узлов в утолщенном торе рода 2, имеющих не более 4 перекрестков. Для этого мы перечисляем графы специального вида и рассматриваем все возможные вложения этих графов в тор рода 2, которые приводят к примарым проекциям. Здесь мы доказываем несколько вспомогательных утверждений, сокращающих перечисление таких вложений. Наконец, на третьем этапе, мы доказываем, что все полученные проекции неэквивалентны. Ряд известных и новых приемов позволил удержать процесс в разумных пределах и строго теоретически доказать полноту построенной таблицы.

Ключевые слова: примарная проекция; узел; утолщенный тор рода 2; таблица.

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