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OPTIMAL CONTROL OVER SOLUTIONS OF A MULTICOMPONENT MODEL OF REACTION-DIFFUSION IN A TUBULAR REACTOR

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This article studies a mathematical model of reaction-diffusion in a tubular reactor based on degenerate equations of reaction-diffusion type defined on a geometric graph. It is precisely the degenerate case that is studied, since when building the mathematical model it is taken into account that the speed of one sought function is significantly higher than the speed of the other. This model belongs to a wide class of semilinear Sobolev-type equations. We give sufficient conditions for the simplicity of the phase manifold of the abstract Sobolev-type equation in the case of *s*-monotone and *p*-coercive operator; we prove the existence and uniqueness of a solution to the Showalter–Sidorov problem in the weak generalized sense, and the existence of optimal control over weak generalized solutions to this problem. On the basis of the abstract theory, we find sufficient conditions for the existence of optimal control for a mathematical model of neural signal transmission.

Keywords: Sobolev-type equations; phase manifold; Showalter–Sidorov problem; reaction-diffusion equations; optimal control problem.

Introduction

Take a finite connected oriented graph $\mathbf{G} = \mathbf{G}(V; E)$ with vertex set $V = \{V_i\}_{i=1}^{M}$ and edge set $E=\{E_j\}_{j=1}^{K}$, where each edge is of length $l_j > 0$ and transverse cross-section area $d_j > 0$. Consider on $\mathbf{G}$ the multicomponent system of reaction-diffusion equations

$$\begin{cases} v_{1jt} = \alpha_1 v_{1jss} + f_{1j}(v_{1j}, v_{2j}, \ldots, v_{mj}) + u_{1j}, \\ v_{2jt} = \alpha_2 v_{2jss} + f_{2j}(v_{1j}, v_{2j}, \ldots, v_{mj}) + u_{2j}, \\ \ldots \\ v_{kjt} = \alpha_k v_{kjss} + f_{kj}(v_{1j}, v_{2j}, \ldots, v_{mj}) + u_{kj}, \\ 0 = \alpha_{k+1} v_{(k+1)jss} + f_{(k+1)j}(v_{1j}, v_{2j}, \ldots, v_{mj}) + u_{(k+1)j}, \\ \ldots \\ 0 = \alpha_m v_{mjss} + f_{mj}(v_{1j}, v_{2j}, \ldots, v_{mj}) + u_{mj} \\ \text{for all } s \in (0, l_j), t \in R, j = \overline{1, K}, \end{cases} \tag{1}$$

with positive parameters α_i, $i = \overline{1,m}$ and some functions $f_{ij} \in C^{\infty}(\mathbb{R}^m, \mathbb{R})$ for $i = \overline{1,m}$ and $j = \overline{1,K}$. Here the functions $v_i = v_i(s, t)$, $i = \overline{1,k}$ and $v_i = v_i(s, t)$, $i = \overline{k+1,m}$ characterize the concentrations of reagents (activator and inhibitor); α_i, $i = \overline{1,m}$ are the diffusion coefficients; the functions f_i correspond to the interaction between the reagents; the prescribed functions $u_i = u_i(s, t)$ characterize exterior actions. For (1) at each vertex V_i for $i = \overline{1,m}$ impose flow balance and continuity conditions

$$\sum_{j:E_j \in E^{\alpha}(V_i)} d_j v_{js}(0,t) - \sum_{r:E_r \in E^{\omega}(V_i)} d_r v_{rs}(l_r, t) = 0, \tag{2}$$

$$v_r(0,t) = v_j(0,t) = v_h(l_h, t) = v_n(l_n, t) \tag{3}$$

for all $E_r, E_j \in E^{\alpha}(V_i)$ and $E_h, E_n \in E^{\omega}(V_i)$. Here $E^{\alpha(\omega)}(V_i)$ stands for the set of edges starting (ending) at V_i. Conditions (2), (3) and the system (1) constitute our mathematical model of reaction-diffusion in a tubular reactor. Complement (2), (3) with the Showalter–Sidorov initial conditions

$$v_{ij}(s,0) = v_{0ij}(s) \text{ for all } s \in (0, l_j), i = \overline{1,k}, j = \overline{1,K}. \tag{4}$$

Initially, a nondegenerate system of equations of the reaction-diffusion type

$$\begin{cases} \varepsilon v_t = \alpha \Delta v + f_1(v,w) + u_1, \\ w_t = \beta \Delta w + f_2(v,w) + u_2. \end{cases} \tag{5}$$

was obtained in [1–3], depending on the two desired functions $v = v(s, t)$ and $w = w(s, t)$. These systems model a large class of processes. In the case

$$f_1(v,w) = \gamma - (\delta + 1)v + v^2 w, \quad f_2(v,w) = \delta v - v^2 w \tag{6}$$

the system (5) describes the Lefever–Prigogine Brusselator [1], proposed as a model of an autocatalytic reaction with diffusion. The FitzHugh–Nagumo model [2, 3] is of this type with

$$f_1(v,w) = \beta_1 w - \kappa_1 v, \quad f_2(v,w) = \beta_2 w - \kappa_2 v - w^3. \tag{7}$$

The first qualitative analysis of the system (5) appeared in [4] under the assumption that the rate of change of one concentration is much greater than that of the other. This assumption leads to the degenerate system

$$\begin{cases} 0 = \alpha \Delta v + f_1(v,w) + u_1, \\ w_t = \beta \Delta w + f_2(v,w) + u_2. \end{cases} \tag{8}$$

The analysis of the morphology [4] of the phase spaces of the degenerate FitzHugh–Nagumo model (7), (8) and the Lefever–Prigogine Brusselator (6), (8) on open bounded regions showed that these phase spaces contain fold and cusp singularities [4]. Multicomponent reaction-diffusion models are studied in [5, 6]. They usually involve many inhibitors. Models with three and four components, one activator and two or three inhibitors, and their stability were studied in [5]. Goal of this article is reseach of multicomponent reaction-diffusuin with different numbers of inhibitors and activators not only inhibitors.

Consider two Banach spaces $\boldsymbol{X}$ and $\boldsymbol{U}$. The preimage of the degenerate system (1) with conditions (2), (3) is the abstract semilinear Sobolev-type equation

$$\frac{d}{dt} Lx + M(x) = u, \ker L \neq \{0\}. \tag{9}$$

Here L is a continuous linear operator and M is a smooth nonlinear operator to be specified. The analytic and qualitative aspects of initial (multipoint initial-final) value problems for linear and semilinear Sobolev-type models are studied in [7–12]. Complement (9) with the Showalter–Sidorov initial condition

$$L(x(0) - x_0) = 0. \tag{10}$$

Considering this initial condition instead of the classical Cauchy condition

$$x(0) = x_0 \tag{11}$$

in the case of degenerate equation (9), we can avoid the lack of existence of a solution for arbitrary initial data [8]. Condition (10) directly generalizes condition (11) since Cauchy and Showalter–Sidorov problems are equivalent in the case that L^{-1} exists and is continuous. However, condition (10) fails to guarantee the uniqueness of solution to problem (9), (10), for instance in the cases that the phase manifold of (9) lies in a Banach manifold with singularities [4, 8]. Thus, to find conditions under which the solution is unique, we must study the structure of the phase manifold.

Our goal is to study the optimal control problem

$$J(x,u) \to \min \tag{12}$$

by the solutions to (9), (10) in the weak generalized sense [13, 14]. Here $J(x,u)$ is a certain purpose-built quality functional with control $u \in \boldsymbol{U}_{ad}$, where $\boldsymbol{U}_{ad}$ is a closed convex set in the control space $\boldsymbol{U}$. The optimal control problem for linear Sobolev-type equations with the Cauchy initial condition was originally posed and studied in [9]. That article initiated a series of studies of optimal control problems for linear Sobolev-type equations with various initial conditions [10–12]. Sufficient conditions for the existence of a solution to problem (9), (10), (12) when L is a Fredholm operator were obtained in [11]. We give sufficient conditions for the simplicity of the phase manifold of problem (1)–(3) in case L is not Fredholm operator. Optimal control problems in various reaction-diffusion models are studied in [12]. The Showalter–Sidorov problem and the optimal control problem for degenerate two-component FitzHugh–Nagumo model (7), (8) is considered in [12] in the case that $\beta_2 \le 0$ and $\beta_1 = \kappa_2$.

This article is organized as follows. In the first section we talk about abstract semilinear Sobolev-type equations and discuss sufficient conditions for the simplicity of the phase manifold of the abstract equations (9) in the case of s-monotone and p-coercive operator M and prove the existence and uniqueness of a solution to (9), (10) in the weak generalized sense using the Galerkin method. In the second section we construct a mathematical model of reaction-diffusion in a tubular reactor basing on the initial-boundary value problem for degenerate reaction-diffusion equations defined on a geometric graph. In the third section we study the optimal control problem for a mathematical model of neural signal transmission and give sufficient conditions for the existence of a solution to it.

1. Abstract semilinear Sobolev-type equation in the case of s-monotone and p-coercive operator

Consider abstract semilinear Sobolev-type equation (9) with the Showalter–Sidorov initial condition (10). All our arguments in this section will be based on the general theory of abstract Sobolev-type equations, which is described in sufficient detail in [8, 11]. Take a separable real Hilbert space $\mathbf{H} = (\mathbf{H};[\cdot,\cdot])$ identified with its adjoint, as well as an adjoint pair $(A; A^*)$ with respect to $[\cdot,\cdot]$ of reflexive separable Banach spaces such that the embeddings

$$A \subset \mathbf{H} \subset A^* \tag{13}$$

are dense and continuous. Take a selfadjoint nonnegative definite operator $L \in L(A;A^*)$ with

$$\mathbf{H} \supset \ker L \equiv \operatorname{coker} L \subset \mathbf{H}^*,\ A = \ker L \oplus \operatorname{coim} L,\ A^* = \operatorname{coker} L \oplus \operatorname{im} L. \tag{14}$$

Remark 1. Condition (14) is satisfied, for instance, in the case that $L \in L(A; A^*)$ is a selfadjoint nonnegative definite Fredholm operator [11].

Take an s-monotone and p-coercive operator $M \in C^r(A; A^*)$ with $r \geq 1$ (that is, $[M'_y x,x]>0$ $\forall x,y \in A\setminus\{0\}$ and $\exists$ C_M, $C^M \in \mathbb{R}_+$ such that $[M(x), x] \geq C_M \|x\|^p$ and $\|M(x)\|_* \leq C^M \|x\|^{p-1}$, where $p \geq 2$) possessing symmetric Fréchet derivative. Note that every strongly monotone operator is s-monotone, while every s-monotone operator is strictly monotone. In turn, every p-coercive operator is strongly coercive.

By condition (14), there exists a projection Q along coker L onto im L. Make the assumption that

$$(I-Q)u \text{ is independent of } t \in (0,T). \tag{15}$$

Consider the set

$$\mathrm{M} = \begin{cases} \{x \in A : (I-Q)Mx = (I-Q)u\}, & if\ \ker L \neq \{0\}; \\ A, & if\ \ker L = \{0\}. \end{cases} \tag{16}$$

Introduce

$$\operatorname{coim} L = \{x \in A : [x,\varphi] = 0\, \forall \varphi \in \ker L \setminus \{0\}\}.$$

Denote by P the projection along ker L onto coim L. Given a point $x_0 \in \boldsymbol{M}$, put $x_0^1 = Px_0 \in$ coim L.

Definition 1. [8] Call a set $\boldsymbol{M}$ a Banach C^r-manifold at $x_0 \in \boldsymbol{M}$ whenever there exist neighborhoods $\mathbf{O} \subset \boldsymbol{M}$ and $\mathbf{O}_1 \subset$ coim L of the points x_0 and $x_0^1 = Px_0$ respectively and a C^r-diffeomorphism D: $\mathbf{O}_1 \to \mathbf{O}$ such that D^{-1} is the restriction of the projection P to $\mathbf{O}$. Refer to the pair $(D, \mathbf{O}_1)$ as a chart. The set $\boldsymbol{M}$ is called a Banach C^r-manifold modeled on the space coim L whenever each of its points admits a chart.

Theorem 1. [8] Suppose that condition (15) is met and M is s-monotone and p-coercive operator. Then the set $\boldsymbol{M}$ is a Banach C^r-manifold projecting diffeomorphically along ker L onto coim L everywhere with the possible exception of the origin.

The proof of the Theorem 1 is analogous to the proof of Theorem 1 in [8].

Remark 2. Observe that if $x = x(t)$ for $t \in [0,T]$ is a solution to (9) then it must lie in $\boldsymbol{M}$. Refer to $\boldsymbol{M}$ as the phase manifold of equation (9).

Since the space A is separable, there is an orthonormal system (in the sense of $\mathbf{H}$) of functions $\{\varphi_i\}$ which is complete in A. Construct Galerkin approximations to the solution to (9), (10) as

$$x^n(s,t) = \sum_{i=1}^{n} a_i(t)\varphi_i(s), \tag{17}$$

where the coefficients $a_i = a_i(t)$ for $i = 1,\ldots,n$ are determined from the following problem:

$$\left[Lx_t^n,\varphi_i\right]+\left[M(x^n),\varphi_i\right]=\left[u,\varphi_i\right], \tag{18}$$

$$\left[L(x^n(0)-x_0),\varphi_i\right]=0, i=1,\ldots,n, \tag{19}$$

$$Lx^n(0)\to Lx_0 \text{ for } n\to+\infty \text{ strongly in the subspace im } L. \tag{20}$$

In the case dimker $L < +\infty$ it is necessary to have $n >$ dimker L. Equation (18) constitute a degenerate system of ordinary differential equations. Suppose that $T_n\in\mathbb{R}_+$, $T_n = T_n(x_0)$, and $A^n = \text{span}\{\varphi_1,\varphi_2,\ldots,\varphi_n\}$.

Lemma 1. [17] Let M be s-monotone and p-coercive operator. For every $x_0\in A$ there exists a unique local solution $x^n\in C^r(0, T_n; A^n)$ to problem (18), (19).

The proof rests on the existence Theorem for solutions to a system of algebraic-differential equations with the Showalter–Sidorov condition [17].

Construct the space

$$\boldsymbol{X} = \{x \mid x\in L_\infty(0,T;\text{coim}\,L)\cap L_p(0,T;A),\ \dot{x}^1\in L_2(0,T;\text{coim } L)\}.$$

Definition 2. [11] Call a weak generalized solution to (9) the vector function $x\in\boldsymbol{X}$ satisfying the condition

$$\int_0^T\varphi(t)\left[\frac{d}{dt}Lx+M(x),w\right]dt=\int_0^T\varphi(t)\left[u,w\right]dt, \forall w\in A, \forall\varphi\in L_2(0,T). \tag{21}$$

Call a solution to (9) a solution to the Showalter–Sidorov problem whenever it satisfies (10).

Theorem 2. [11] Let M be s-monotone and p-coercive operator. For every $x_0\in A$, $T\in\mathbb{R}_+$, $u\in L_2(0, T; A^*)$ there exists a unique solution $x\in\boldsymbol{X}$ to problem (9), (10).

This goes in several stages and relies on the monotonicity method of [13, 14].

Assume that all requirements of the previous section are satisfied and the embedding $A\subset\mathbf{H}$ is compact. Construct the space $\boldsymbol{U} = L_q(0, T; A^*)$, $\frac{1}{p}+\frac{1}{q}=1$ and define in it a nonempty closed convex set $\boldsymbol{U}_{ad}$. Consider the optimal control problem

$$J(x,u)\to\inf,\ \ u\in\boldsymbol{U}_{ad} \tag{22}$$

defining the objective functional as

$$J(x,u)=\beta\int_0^T\|x(t)-z_d(t)\|_A^p\,dt+(1-\beta)\int_0^T\|u(t)\|_{A^*}^q\,dt,\ \beta\in(0,1), \tag{23}$$

Here $z_d = z_d(t)$ is the required state.

Definition 3. [11] Refer to a pair $(\tilde{x},\tilde{u})\in\boldsymbol{X}\times\boldsymbol{U}_{ad}$ as a solution to the optimal control problem (9), (10), (22) if

$$J(\tilde{x},\tilde{u})=\inf_{(x,u)}J(x,u),$$

where the pairs $(x, u)\in$ X×$\boldsymbol{X}\times\boldsymbol{U}_{ad}$ satisfy (9), (10) in the sense of Definition 2; call the vector function $\tilde{u}$ the optimal control.

Remark 3. Refer as an admissible element of problem (9), (10), (23) to a pair $(x,u)\in\boldsymbol{X}\times\boldsymbol{U}_{ad}$, satisfying problem (9), (10) with

$$J(x,u)<+\infty.$$

If $\boldsymbol{U}_{ad}\neq\varnothing$ then for every $u\in\boldsymbol{U}_{ad}\subset\boldsymbol{U}$ by Theorem 2 there exists a unique solution $x = x(u)$ to problem (9), (10). Hence, the set of admissible elements of the problem is nonempty. Using the results obtained in the paper [11] we can show that

Theorem 3. [11] Let M be s-monotone and p-coercive operator. Given $x_0\in A$ and $T\in\mathbb{R}_+$, there exists a solution to problem (9), (10), (22).

2. A mathematical model of reaction-diffusion in a tubular reactor

In this section we construct a mathematical model of reaction-diffusion in a tubular reactor basing on the initial-boundary value problem for degenerate reaction-diffusion equations defined on a geometric graph and reduce it to the abstract Showalter–Sidorov problem (9), (10), we construct functional spaces and establish the main properties of operators. Take on finite connected oriented graph **G** the multicomponent system of reaction-diffusion equations (1) with flow balance and continuity conditions (2), (3) and the Showalter–Sidorov initial conditions (4).

Consider the Hilbert space

$$L_2(\mathrm{G}) = \{g = (g_1, g_2, \ldots, g_j, \ldots, g_K) : g_j \in L_2(0, l_j)\}$$

equipped with the inner product

$$\langle g, h\rangle = \sum_{E_j \in E} d_j \int_0^{l_j} g_j(s) h_j(s) ds.$$

Construct the Banach space

$$H = \{g = (g_1, g_2, \ldots, g_j, \ldots, g_K) : g_j \in W_2^1(0, l_j) \text{ and conditions (3) holds}\}$$

with the norm

$$\|g\|_H^2 = \sum_{E_j \in E} d_j \int_0^{l_j} (g_{js}^2(s) + g_j^2(s)) ds.$$

Put

$$L_p(\mathrm{G}) = \{g = (g_1, g_2, \ldots, g_j, \ldots, g_K) : g_j \in L_p(0, l_j)\}.$$

The set L_p(**G**) is a Banach space with the norm

$$\|g\|_{L_p(\mathrm{G})}^p = \sum_{E_j \in E} d_j \int_0^{l_j} |g_j(s)|^p \, ds.$$

By the Sobolev embedding theorem, the space W_2^1 $(0,l_j)$ consists of absolutely continuous functions, and so H is well-defined, dense, and compactly embedded into L_2 (**G**). Fix $a>0$ and construct the operator

$$\langle \mathrm{A} g, h\rangle = \sum_{E_j \in E} d_j \int_0^{l_j} (g_{js}(s) h_{js}(s) + a g_j(s) h_j(s)) ds, \quad g, h \in H.$$

The operator $\mathbf{A} \in L(H;H^*)$ is bijective, its spectrum is real, discrete, of finite multiplicity, and accumulates only at $+\infty$, while its eigenfunctions constitute a basis for the space H [15]. Denote by $\{\varphi_i\}$ a sequence of eigenfunctions of the homogeneous Dirichlet problem for the operator **A** on the graph **G**, and by $\{\mu_i\}$ the associated sequence of eigenvalues in decreasing order with multiplicities taken into account.

Consider the Hilbert space

$$\mathbf{H} = L_2^m(\mathrm{G}) = \{v = (v_1, v_2, \ldots, v_m) : v_i \in L_2(\mathrm{G})\}$$

equipped with the inner product

$$[v, \zeta] = \sum_{i=1}^m \langle v_i, \zeta_i\rangle$$

and identified with its adjoint. By analogy, construct the space $A = H^m$ and denote by A^* the adjoint to A with respect to the inner product in **H**. Writing $x = (v_1, v_2, \ldots, v_m)$, $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_m)$, and $u = (u_1, u_2, \ldots, u_m)$, define the operators

$$[Lx, \zeta] = \langle v_1, \eta_1\rangle + \ldots + \langle v_k, \eta_k\rangle, \quad x, \zeta \in A$$

$$[M(x), \zeta] = \alpha_1 \langle v_{1s}, \zeta_{1s}\rangle + \langle f_1(x), \zeta_1\rangle + \alpha_2 \langle v_{2s}, \zeta_{2s}\rangle + \langle f_2(x), \zeta_2\rangle + \ldots$$
$$+ \alpha_m \langle v_{ms}, \zeta_{ms}\rangle + \langle f_m(x), \zeta_m\rangle, \quad x, \zeta \in A.$$

Lemma 2. (i) The operator $L \in \mathbf{L}(A; A^*)$ is selfadjoint and nonnegative definite.

(ii) Suppose that $f_{ij} \in C^\infty(\mathbb{R}^m, \mathbb{R})$ for $i = \overline{1,m}$ and $j = \overline{1,K}$. Then $M \in C^\infty(A; A^*)$.

Proof. Claim (i) follows from the construction of L. The containment $M \in C^\infty(A; A^*)$ is a classical result [16].

Thus, problem (1)–(4) reduces to the Showalter–Sidorov problem (9), (10).

3. Optimal control problem for a mathematical model of neural signal transmission

In the this section we apply the abstract results of the second section to study the optimal control problem for a mathematical model of neural signal transmission, which can be obtained from a multicomponent reaction-diffusion model (1) if f_i take as (6). Proceed to a mathematical model of neural signal transmission based on the FitzHugh–Nagumo system

$$\begin{cases} v_{1jt} - \alpha_1 v_{1jss} + \beta_{11} v_{1j} + \beta_{12} v_{2j} + \ldots + \beta_{1m} v_{mj} + \kappa_1 v_{1j}^3 = u_{1j}, \\ v_{2jt} - \alpha_2 v_{2jss} + \beta_{21} v_{1j} + \beta_{22} v_{2j} + \ldots + \beta_{2m} v_{mj} + \kappa_2 v_{2j}^3 = u_{2j}, \\ \ldots \\ v_{kjt} - \alpha_k v_{kjss} + \beta_{k1} v_{1j} + \beta_{k2} v_{2j} + \ldots + \beta_{km} v_{mj} + \kappa_k v_{kj}^3 = u_{kj}, \\ -\alpha_{k+1} v_{(k+1)jss} + \beta_{(k+1)1} v_{1j} + \beta_{(k+1)2} v_{2j} + \ldots + \beta_{(k+1)m} v_{mj} = u_{(k+1)j}, \\ \ldots \\ -\alpha_m v_{mjss} + \beta_{m1} v_{1j} + \beta_{m2} v_{2j} + \ldots + \beta_{mm} v_{mj} = u_{mj} \\ \text{for all } s \in (0, l_j), t \in R, j = \overline{1,K}, \end{cases} \quad (24)$$

defined on a finite connected oriented graph **G** and complemented with conditions (2), (3), where the matrix $B = \{\beta_{ij}\}_{i,j=1}^{m}$ has the property

$$\exists C_B, C_B > 0 : C_B [x,x] \le [Bx, x] \le C^B [x,x]. \quad (25)$$

By analogy with Section 2, consider the Hilbert space $\mathbf{H}=(L_2^m(\mathbf{G}), [\cdot,\cdot])$ and the Banach space $A = H^m$. By the Sobolev embedding theorem, there are dense continuous embeddings (13); furthermore, the embedding $A \subset \mathbf{H}$ is compact. Write $x = (v_1, v_2, \ldots, v_m)$, $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_m)$, and $u = (u_1, u_2, \ldots, u_m)$. Then the operator $M = M_1 + M_2$ becomes

$$[M_1(x), \zeta] = \alpha_1 \langle v_{1s}, \zeta_{1s} \rangle + \langle \beta_{11} v_1 + \beta_{12} v_2 + \ldots + \beta_{1m} v_m, \zeta_1 \rangle + \alpha_2 \langle v_{2s}, \zeta_{2s} \rangle +$$
$$+ \langle \beta_{21} v_1 + \beta_{22} v_2 + \ldots + \beta_{2m} v_m, \zeta_2 \rangle + \ldots + \alpha_m \langle v_{ms}, \zeta_{ms} \rangle +$$
$$+ \langle \beta_{m1} v_1 + \beta_{m2} v_2 + \ldots + \beta_{mm} v_m, \zeta_m \rangle, \; x, \zeta \in A,$$
$$[M_2(x), \zeta] = \kappa_1 \langle v_1^3, \zeta_1 \rangle + \kappa_2 \langle v_2^3, \zeta_2 \rangle + \ldots + \kappa_k \langle v_k^3, \zeta_1 \rangle, \; x, \zeta \in A,$$

where $v_m^3 = (v_{m1}^3, v_{m2}^3, \ldots, v_{mk}^3)$.

Lemma 3. (i) Suppose that $\alpha_i \in \mathbb{R}_+$ for $i = \overline{1,m}$ and condition (25) is satisfied. Then the operator $M_1 \in C^\infty(A; A^*)$ is s-monotone and 2-coercive.
(ii) Suppose that $\kappa_i \in \mathbb{R}_+$ for $i = \overline{1,k}$. Then the operator $M_2 \in C^\infty(L_4^k(\mathbf{G}), L_{\frac{4}{3}}^k(\mathbf{G}))$ is s-monotone and 4-coercive.

Proof. The Fréchet derivatives of M_1 and M_2 at $x \in A$ are defined as

$$[M'_{1x}\xi, \zeta] = \alpha_1 \langle \xi_{1s}, \zeta_{1s} \rangle + \langle \beta_{11}\xi_1 + \beta_{12}\xi_2 + \ldots + \beta_{1m}\xi_m, \zeta_1 \rangle +$$
$$+ \alpha_2 \langle \xi_{2s}, \zeta_{2s} \rangle + \langle \beta_{21}\xi_1 + \beta_{22}\xi_2 + \ldots + \beta_{2m}\xi_m, \zeta_2 \rangle + \ldots +$$
$$+ \alpha_m \langle \xi_{ms}, \zeta_{ms} \rangle + \langle \beta_{m1}\xi_1 + \beta_{m2}\xi_2 + \ldots + \beta_{mm}\xi_m, \zeta_m \rangle, \;\; x, \zeta \in A,$$
$$[M'_{2x}\xi, \zeta] = 3\kappa_1 \langle v_1^2 \xi_1, \zeta_1 \rangle + 3\kappa_2 \langle v_2^2 \xi_2, \zeta_2 \rangle + \ldots + 3\kappa_k \langle v_k^2 \xi_k, \zeta_k \rangle, \; x, \zeta \in A.$$

Then the continuous embedding $W_2^1(\mathbf{G}) \subset L_2(\mathbf{G})$ yields

$$|\left[M'_{1x}\xi, \zeta\right]| \le C \|\xi\|_A \|\zeta\|_A,$$
$$|\left[M'_{2x}\xi, \zeta\right]| \le 3C \|\xi\|_A \|x\|_A^2 \|\zeta\|_A,$$
$$|\left[M''_{2x}(\xi_1, \xi_2), \zeta\right]| \le 6C \|\xi_1\|_A \|\xi_2\|_A \|x\|_A \|\zeta\|_A,$$
$$|\left[M'''_{2x}(\xi_1, \xi_2, \xi_3), \zeta\right]| \le 6C \|\xi_1\|_A \|\xi_2\|_A \|\xi_3\|_A \|\zeta\|_A, C = \max_i \kappa_i,$$

where M'_{1x} and M'_{2x} stand for the Fréchet derivatives of M_1 and M_2 at x. Since $M_{1x}^{(2)} \equiv O$ and $M_{2x}^{(4)} \equiv O$, the operators M_1 and M_2 are C^∞-smooth. Since

$$[M_{1x}'\xi,\xi] = \alpha_1\langle\xi_{1s},\xi_{1s}\rangle + \langle\beta_{11}\xi_1 + \beta_{12}\xi_2 + \ldots + \beta_{1m}\xi_m,\xi_1\rangle +$$
$$+\alpha_2\langle\xi_{2s},\xi_{2s}\rangle + \langle\beta_{21}\xi_1 + \beta_{22}\xi_2 + \ldots + \beta_{2m}\xi_m,\xi_2\rangle + \ldots +$$
$$+\alpha_m\langle\xi_{ms},\xi_{ms}\rangle + \langle\beta_{m1}\xi_1 + \beta_{m2}\xi_2 + \ldots + \beta_{mm}\xi_m,\xi_m\rangle > 0,\ \ x,\xi\in A,$$
$$[M_{2x}'\xi,\xi] = 3\kappa_1\langle v_1^2\xi_1,\xi_1\rangle + 3\kappa_2\langle v_2^2\xi_2,\xi_2\rangle + \ldots + 3\kappa_k\langle v_k^2\xi_k,\xi_k\rangle > 0,\ \ x,\xi\in A,$$

it follows that the operators M_1 and M_2 are s -monotone. Since

$$C_1\|x\|_A^2 \le [M_1(x),x] \le C_2\|x\|_A^2,$$
$$[M_2(x),x] = \kappa_1\|v_1\|_{L_4}^4(\mathbf{G}) + \kappa_2\|v_2\|_{L_4}^4(\mathbf{G}) + \ldots + \kappa_k\|v_k\|_{L_4}^4(\mathbf{G}),$$

it follows that M_1 is 2-coercive and M_2 is 4-coercive.

Remark 4. By the construction of L, the sets ker L, coim L, coker L, and im L are defined as

$$\ker L = \underbrace{\{0\}\times\{0\}\times\ldots\times\{0\}}_{k}\underbrace{\times H\times H\times\ldots\times H}_{m-k},$$
$$\operatorname{coim} L = \underbrace{H\times H\times\ldots\times H}_{k}\underbrace{\times\{0\}\times\{0\}\times\ldots\times\{0\}}_{m-k},$$
$$\operatorname{coker} L = \underbrace{\{0\}\times\{0\}\times\ldots\times\{0\}}_{k}\underbrace{\times H^*\times H^*\times\ldots\times H^*}_{m-k},$$
$$\operatorname{im} L = \underbrace{H^*\times H^*\times\ldots\times H^*}_{k}\underbrace{\times\{0\}\times\{0\}\times\ldots\times\{0\}}_{m-k}.$$

Hence, condition (14) is satisfied.

Construct the set

$$\boldsymbol{M} = \Big\{x\in A: \langle\alpha_{k+1}v_{(k+1)s},\xi_{(k+1)s}\rangle + \langle\beta_{(k+1)1}v_1 + \beta_{(k+1)2}v_2 + \ldots +$$
$$+\beta_{(k+1)m}v_m,\xi_{k+1}\rangle + \ldots + \langle\alpha_m v_{ms},\xi_{ms}\rangle + \langle\beta_{m1}v_1 + \beta_{m2}v_2 + \ldots +$$
$$+\beta_{mm}v_m,\xi_m\rangle = \langle u_{(k+1)},\xi_{k+1}\rangle + \ldots + \langle u_m,\xi_m\rangle\Big\}.$$

Condition (15) becomes

$$(0,\ldots,0,u_{k+1},\ldots,u_m) \text{ is independent of } t\in(0,T) \tag{26}$$

Then Theorem 1 and Lemmas 2 and 3 imply the following theorem.

Theorem 4. Suppose that $\alpha_i\in\mathbb{R}_+$ for $i=\overline{1,m}$, $\kappa_i\in\mathbb{R}_+$ for $i=\overline{1,k}$ and conditions (25) and (26) are satisfied. Then the set $\boldsymbol{M}$ is a simple Banach C^∞-manifold modeled on the subspace coim L.

Construct the spaces

$$\boldsymbol{X} = \{x = (v_1,v_2,...,v_k,v_{k+1},...,v_m): v_i\in L_\infty(0,T;H)\cap L_4(0,T;H)),$$
$$\frac{dv_i}{dt}\in L_2(0,T;H), i=1,...,k;\ v_i\in L_\infty(0,T;H)\cap L_2(0,T;H)),$$
$$\boldsymbol{U} = \{u = (u_1,u_2,...,u_m): u_i\in L_{\frac{4}{3}}(0,T;L_4(G)), i=1,\ldots,k;$$
$$u_i\in L_2(0,T;H^*), i=k+1,...,m\}.$$

By analogy with Section 2, construct an orthonormal system $\{\varphi_1, \varphi_2,\ldots, \varphi_i\}$, where $\{\varphi_i\}$ are eigenvectors of A, which in view of the embedding (13) constitutes a basis for the space $\mathbf{H}$. Construct Galerkin approximations to the solution to problem (2), (3), (24) as

$$v_i^n(s,t) = \sum_{l=1}^n a_i^l(t)\varphi_l(s),\ i=1,\ldots,m,$$

where the coefficients a_i^l for $i=\overline{1,m}$ and $l=\overline{1,n}$ are determined by the system

$$\begin{cases} \langle v_{1t}^n - \alpha_1 v_{1ss}^n + \beta_{11} v_1^n + \beta_{12} v_2^n + \ldots + \beta_{1m} v_m^n + \kappa_1 (v_1^n)^3, \varphi_i \rangle = \langle u_1, \varphi_i \rangle, \\ \langle v_{2t}^n - \alpha_2 v_{2ss}^n + \beta_{21} v_1^n + \beta_{22} v_2^n + \ldots + \beta_{2m} v_m^n + \kappa_2 (v_2^n)^3, \varphi_i \rangle = \langle u_2, \varphi_i \rangle, \\ \ldots \\ \langle v_{kt}^n - \alpha_k v_{kss}^n + \beta_{k1} v_1^n + \beta_{k2} v_2^n + \ldots + \beta_{km} v_m^n + \kappa_k (v_k^n)^3, \varphi_i \rangle = \langle u_k, \varphi_i \rangle, \\ \langle -\alpha_{k+1} v_{(k+1)ss}^n + \beta_{(k+1)1} v_1^n + \beta_{(k+1)2} v_2^n + \ldots + \beta_{(k+1)m} v_m^n, \varphi_i \rangle = \langle u_{(k+1)}, \varphi_i \rangle, \\ \ldots \\ \langle -\alpha_m v_{mss}^n + \beta_{m1} v_1^n + \beta_{m2} v_2^n + \ldots + \beta_{mm} v_m^n, \varphi_i \rangle = \langle u_m, \varphi_i \rangle, \\ i = \overline{1,n}, \end{cases}$$

and the Showalter–Sidorov conditions

$$\langle v_1(0) - v_{01}, \varphi_i \rangle = 0, \ldots, \langle v_k(0) - v_{0k}, \varphi_i \rangle = 0, \; i = \overline{1,n}.$$

Then Theorem 2 and Lemmas 2 and 3 imply the following theorem.

Theorem 5. Suppose that $\alpha_i \in \mathbb{R}_+$ for $i = \overline{1,m}$, $\kappa_i \in \mathbb{R}_+$ for $i = \overline{1,k}$ and conditions (25) are satisfied. Given $x_0 \in A$ and $u \in \boldsymbol{U}$, there exists a unique solution $x \in \boldsymbol{X}$ to problem (2)–(4), (24).

Choose a nonempty closed convex set $\boldsymbol{U}_{ad} \subset \boldsymbol{U}$. Consider the optimal control problem

$$J(x,u) \to \inf \tag{27}$$

by solutions to problem (2)–(4), (25), where the objective functional is defined as

$$J(x,u) = \beta \sum_{i=1}^{k} \int_0^T \left\| v_i - z_i^d \right\|_{L_2(\mathrm{G})}^2 dt + \beta \sum_{i=k+1}^{m} \int_0^T \left\| v_i - z_i^d \right\|_{L_4(\mathrm{G})}^4 dt +$$

$$+ (1-\beta) \sum_{i=1}^{k} \int_0^T \left\| u_i \right\|_{L_2(\mathrm{G})}^2 dt + (1-\beta) \sum_{i=k+1}^{m} \int_0^T \left\| u_m \right\|_{L_{\frac{4}{3}}(\mathrm{G})}^{\frac{4}{3}} dt, \beta \in (0,1),$$

Then Theorem 3 and Lemmas 2 and 3 imply the following Theorem.

Theorem 6. Suppose that α_i, $\kappa_i \in \mathbb{R}_+$ for $i = \overline{1,m}$ and conditions (26) are satisfied. Then for every $x_0 \in A$ problem (2)–(4), (24), (27) admits optimal control.

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References

1. Prigogine I., Lefever R. Symmetry breaking instabilities in dissipative systems II. *The Journal of Chemical Physics*, 1968, Vol. 48, Iss. 4, pp. 1695–1700. DOI: 10.1063/1.1668896

2. FitzHugh R. Impulses and Physiological States in Theoretical Models of Nerve Membrane. *Biophysical Journal*, 1961, Vol. 1, no. 6, pp. 445–466. DOI: 10.1016/s0006-3495(61)86902-6

3. Nagumo J., Arimoto S., Yoshizawa S. An active pulse transmission line simulating nerve axon. *Proceedings of the IRE*, 1962, Vol. 50, Iss. 10, pp. 2061–2070. DOI: 10.1109/JRPROC.1962.288235

4. Bokareva T.A, Sviridiuk G.A. Whitney assemblies of phase spaces of certain semilinear equations of Sobolev type. *Mathematical Notes*, 1994, Vol. 55, no. 3, pp. 237–242. DOI: 10.1007/BF02110776

5. Gubernov V.V., Kolobov A.V., Polezhaev A.A., Sidhu H.S., McIntosh A.C., Brindley J. Stabilization of combustion wave through the competitive endothermic reaction. *Proceeding of the Royal Society A*, 2015, Vol. 471, no. 2180, pp. 20150293. DOI: 10.1098/rspa.2015.0293

6. Savchik J., Chang Br., Rabitz H. Application of moments to the general linear multicomponent reaction-diffusion equation. *Journal of Physical Chemistry,* 1983, Vol. 87, no. 11, pp. 1990–1997. DOI: 10.1021/j100234a031

7. Zagrebina S.A. A Multipoint Initial-Final Value Problem for a Linear Model of Plane-Parallel Thermal Convection in Viscoelastic Incompressible Fluid. *Bulletin of the South Ural State University. Series: Mathematical Modelling, Programming and Computer Software*, 2014, Vol. 7, no. 3, pp. 5–22. DOI: 10.14529/mmp140301

8. Sviridyuk G.A., Manakova N.A. The Phase Space of the Cauchy–Dirichlet Problem for the Oskolkov Equation of Nonlinear Filtration. *Russian Mathematics*, 2003, Vol. 47, no. 9, pp. 33–38.

9. Sviridyuk G.A., Efremov A.A. Optimal Control of Sobolev Type Linear Equations with Relativity *p*-Sectorial Operators. *Differential Equations*, 1995, Vol. 31, no. 11, pp. 1882–1890.

10. Zamyshlyaeva A.A., Tsyplenkova O.N., Bychkov E.V. Optimal Control of Solutions to the Initial-Final for the Sobolev Type Equation of Higher Order. *Journal of Computational and Engineering Mathematics*, 2016, Vol. 3, no. 2, pp. 57–67. DOI: 10.14529/jcem1602007

11. Sviridyuk G.A., Manakova N.A. An optimal control problem for the Hoff equation. *Journal of Applied and Industrial Mathematics*, 2007, Vol. 1, no. 2, pp. 247–253.

12. Manakova N.A., Gavrilova O.V. Optimal Control for a Mathematical Model of Nerve Impulse Spreading. *Bulletin of the South Ural State University. Ser. Mathematical Modelling, Programming & Computer Software*, 2015, Vol. 8, no. 4, pp. 120–126. DOI: 10.14529/mmp150411

13. Al'shin A.B., Korpusov M.O., Sveshnikov A.G. *Blow-up in nonlinear sobolev-type equations*. Berlin, N.Y., Walter de Gruyter GmbH and Co. KG, 2011, 648 p. DOI: 10.1515/9783110255294

14. Lions J.-L. *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Paris, Dunod, 1969, 554 p. (in French).

15. Bayazitova A.A. The Sturm–Liouville problem on geometric graph. *Bulletin of the South Ural State University. Series: Mathematical Modelling, Programming and Computer Software*, 2010, no. 16 (192), Issue 5, pp. 4–10. (in Russ.).

16. Hassard D.D., Kazarinoff N.D., Wan Y.-H. *Theory and Applications of Hopf Bifurcation*. Cambridge, New York, Cambridge University Press, 1981, 311 p.

17. Sviridyuk G.A. O razreshimosti singulyarnoy sistemy obyknovennykh differentsial'nykh uravneniy (On the Solvability of Singular Systems of Ordinary Differential Equations). *Differentsial'nye Uravneniya*, 1987, Vol. 23, no. 9, pp. 1637–1639. (in Russ.).

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ОПТИМАЛЬНОЕ УПРАВЛЕНИЕ РЕШЕНИЯМИ МНОГОКОМПОНЕНТНОЙ МОДЕЛИ РЕАКЦИИ-ДИФФУЗИИ В ТРУБЧАТОМ РЕАКТОРЕ

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Статья посвящена изучению математической модели реакции-диффузии в трубчатом реакторе на основе вырожденных уравнений типа реакции-диффузии, заданных на геометрическом графе. Исследуется именно вырожденный случай, так как при построении математической модели учитывается, что скорость одной искомой функции значительно превышает скорость другой. Изучаемая модель относится к широкому классу полулинейных моделей соболевского типа. Приводятся достаточные условия простоты фазового многообразия абстрактного уравнения соболевского типа в случае *s*-монотонного и *p*-коэрцитивного оператора; доказываются существование и единственность решения задачи Шоуолтера–Сидорова в слабом обобщенном смысле и существование оптимального управления слабыми обобщенными решениями рассматриваемой задачи. На основе абстрактной теории найдены достаточные условия существования оптимального управления для математической модели передачи импульса по нейронам.

Ключевые слова: уравнения соболевского типа; фазовое многообразие; задача Шоуолтера–Сидорова; система уравнений реакция-диффузия; задача оптимального управления.

Литература

1. Prigogine, I. Symmetry breaking instabilities in dissipative systems II / I. Prigogine, R. Lefever // The Journal of Chemical Physics. – 1968. – Vol. 48, Iss. 4. – P. 1695–1700.

2. FitzHugh, R. Impulses and physiological states in theoretical models of nerve membrane / R. FitzHugh // Biophysical Journal. – 1961. – Vol. 1, no. 6. – P. 445–466.

3. Nagumo, J. An active pulse transmission line simulating nerve axon / J. Nagumo, S. Arimoto, S. Yoshizawa // Proceedings of the IRE. – 1962. – Vol. 50, Iss. 10. – P. 2061–2070.

4. Бокарева, Т.А. Сборки Уитни фазовых пространств некоторых полулинейных уравнений типа Соболева / Т.А. Бокарева, Г.А. Свиридюк // Математические заметки. – 1994. – Т. 55, № 3. – С. 3–10.

5. Stabilization of combustion wave through the competitive endothermic reaction / V.V. Gubernov, A.V. Kolobov, A.A. Polezhaev *et al.* // Proceeding of the Royal Society A. – 2015. – Vol. 471, no. 2180. – P. 20150293.

6. Savchik, J. Application of moments to the general linear multicomponent reaction-diffusion equation / J. Savchik, Br. Chang, H. Rabitz // Journal of Physical Chemistry. – 1983. – Vol. 87, no. 11. – P. 1990–1997.

7. Zagrebina, S.A. A Multipoint Initial-Final Value Problem for a Linear Model of Plane-Parallel Thermal Convection in Viscoelastic Incompressible Fluid / S.A. Zagrebina // Вестник ЮУрГУ. Серия «Математическое моделирование и программирование». – 2014. – Т. 7, № 3. – P. 5–22.

8. Свиридюк, Г.А. Фазовое пространство задачи Коши – Дирихле для уравнения Осколкова нелинейной фильтрации / Г.А. Свиридюк, Н.А. Манакова // Известия вузов. Серия: Математика. –2003. – № 9. – С. 36–41.

9. Свиридюк, Г.А. Оптимальное управление линейными уравнениями типа Соболева с относительно *p*-секториальными операторами / Г.А. Свиридюк, А.А. Ефремов // Дифференциальные уравнения. – 1995. – Т. 31, № 11. – С. 1912–1919.

10. Zamyshlyaeva, A.A. Optimal control of solutions to the initial-final for the Sobolev type equation of higher order / A.A. Zamyshlyaeva, O.N. Tsyplenkova, E.V. Bychkov // Journal of Computational and Engineering Mathematics. – 2016. – Vol. 3, no. 2. – P. 57–67.

11. Свиридюк, Г.А. Задача оптимального управления для уравнения Хоффа / Г.А. Свиридюк, Н.А. Манакова, // Сибирский журнал индустриальной математики. – 2005. – Т. 8, № 2. – С. 144–151.

12. Манакова, Н.А. Оптимальное управление для одной математической модели распростренения нервного импульса / Н.А. Манакова, О.В. Гаврилова // Вестник ЮУрГУ. Серия «Математическое моделирование и программирование». – 2015. – Т. 8, № 4. – С. 120–126.

13. Al'shin, A.B. Blow-up in Nonlinear Sobolev-type Equations / A.B. Al'shin, M.O. Korpusov, A.G. Sveshnikov. – Berlin; N.Y.: Walter de Gruyter GmbH and Co. KG, 2011. – 648 p.

14. Лионс, Ж.-Л. Некоторые методы решения нелинейных краевых задач / Ж.-Л. Лионс. – М.: Мир, 1972. – 588 p.

15. Баязитова, А.А. Задача Штурма–Лиувилля на геометрическом графе / А.А. Баязитова // Вестник ЮУрГУ. Серия «Математическое моделирование и программирование». – 2010. – Вып. 5. – № 16 (192). – С. 4–10.

16. Хэссард, Б. Теория и приложения бифуркации рождения цикла / Б. Хэссард, Н. Казаринов, И. Вэн. – М.: Мир, 1985. – 280 с.

17. Свиридюк, Г.А. О разрешимости сингулярной системы обыкновенных дифференциальных уравнений / Г.А. Свиридюк // Дифференциальные уравнения. – 1987. – Т. 23, № 9. – С. 1637–1639.

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