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ON BASIS PROPERTY OF ROOT FUNCTIONS FOR A CLASS OF THE SECOND ORDER DIFFERENTIAL OPERATORS

V. Ala, Kh.R. Mamedov
Mersin University, Mersin, Turkey
E-mail: volkanala@mersin.edu.tr, hanlar@mersin.edu.tr

It is well known that the Sturmian theory is an important tool in solving numerous problems of mathematical physics. Usually, eigenvalue parameter appears linearly only in the differential equation of the classic Sturm–Liouville problems. However, in mathematical physics there are also problems, which contain eigenvalue parameter not only in differential equation, but also in the boundary conditions.

In this paper, we consider a Sturm–Liouville equation with the eigenparameter dependent boundary condition and with transmission conditions at two points of discontinuity. The aim of this paper is to investigate the completeness, minimality and basis properties of rootfunctions for the considered boundary value problem.

Keywords: eigenfunctions; orthonormal basis; Riesz basis; completeness.

Introduction

In this work, we consider

$$\ell(u) :\equiv -u'' + q(x)u = \lambda u, \tag{1}$$

for $x \in [-1, \xi_1) \cup (\xi_1, \xi_2) \cup (\xi_2, 1]$ with the boundary conditions

$$L_1(u) := \alpha_1 u(-1) + \alpha_2 u'(-1) = 0, \tag{2}$$

$$L_2(u) := \left(\beta_1 u(1) - \beta_2 u'(1)\right) + \lambda\left(\tilde{\beta}_1 u(1) - \tilde{\beta}_2 u'(1)\right) = 0 \tag{3}$$

and the transmission conditions

$$L_3(u) := u(\xi_1 - 0) - \delta u(\xi_1 + 0) = 0, \tag{4}$$

$$L_4(u) := u'(\xi_1 - 0) - \delta u'(\xi_1 + 0) = 0, \tag{5}$$

$$L_5(u) := u(\xi_2 - 0) - \gamma u(\xi_2 + 0) = 0, \tag{6}$$

$$L_6(u) := u'(\xi_2 - 0) - \gamma u'(\xi_2 + 0) = 0, \tag{7}$$

where $-1 < \xi_1 < \xi_2 < 1$, $q(x)$ is a real-valued function, which is continuous on $[-1, \xi_1) \cup (\xi_1, \xi_2) \cup (\xi_2, 1]$ and has finite limits $q(\xi_i \pm 0) := \lim\limits_{x \to \xi_i \pm 0} q(x)$ $(i = 1, 2)$; λ is a complex parameter; δ and γ are positive coefficients, $\alpha_j, \tilde{\alpha}_j, \beta_j, \tilde{\beta}_j$ $(j = 1, 2)$ are real numbers such that $|\alpha_1| + |\alpha_2| \neq 0, |\beta_1| + |\beta_2| \neq 0$, $|\tilde{\beta}_1| + |\tilde{\beta}_2| \neq 0$ and $\rho := \beta_1\tilde{\beta}_2 - \tilde{\beta}_1\beta_2 \neq 0$. In [16], the asymptotic formulas for the eigenvalues and eigenfunctions of problem (1)–(7) are obtained.

Spectral problems for Sturm–Liouville equations with the eigenparameter dependent boundary conditions are of particular interest due to physical applications and are examined in [2, 6, 9, 10, 17]. To this end, the method of separation of variables is applied to solve the corresponding partial differential equation when the boundary conditions contain a directional derivative. Problems on eigenvalue for the second order equation with spectral parameter in the boundary conditions are considered in [5, 7, 8, 11–15, 18]. The corresponding problems led to the eigenvalue problem for a linear operator acting on the space $L_2 \oplus \mathbb{C}^N$, where $\mathbb{C}^N$ is $N-$ dimensional Euclidean space of complex numbers. In [9], for distinct cases, it is shown that the eigenfunctions of the spectral problem formed a defect basis in $L_2(0,1)$. In [4], Rayleigh–Ritz formula is developed for eigenvalues.

The goal of this work is to investigate the problem of completeness, minimality and basis property of the eigenfunctions of boundary value problem (1)–(7). In this study, we introduce a special inner product in a special Hilbert space and construct a linear operator A in the space such that problem (1)–(7) can be interpreted as the eigenvalue problem for A.

1. Operator Theoretic Formulation of the Problem

In this section, we introduce a special inner product in the Hilbert Space $H = L_2[-1,1] \oplus \mathbb{C}$ and define a linear operator A in the space such that problem (1)–(7) can be interpreted as the eigenvalue problem for A.

For $\rho > 0$, let us define the inner product in H by

$$\left(\tilde{u},\tilde{v}\right) = \int_{-1}^{\xi_1} u(x)\overline{v(x)}dx + \delta^2 \int_{\xi_1}^{\xi_2} u(x)\overline{v(x)}dx + \delta^2\gamma^2 \int_{\xi_2}^{1} u(x)\overline{v(x)}dx + \frac{\delta^2\gamma^2}{\rho} u_1\overline{v_1} \tag{8}$$

for

$$\tilde{u} = \begin{pmatrix} u(x) \\ u_1 \end{pmatrix}, \tilde{v} = \begin{pmatrix} v(x) \\ v_1 \end{pmatrix} \in H.$$

For convenience, we use the notations

$$R_1(u) = \beta_1 u(1) - \beta_2 u'(1),$$
$$\tilde{R}_1(u) = \tilde{\beta}_1 u(1) - \tilde{\beta}_2 u'(1).$$

In this Hilbert space, we construct the operator $A: H \to H$ as

$$A\tilde{u} = \begin{pmatrix} -u'' + q(x)u \\ -R_1(u) \end{pmatrix}$$

on the domain

$$D(A) = \left\{ \begin{array}{c} \tilde{u} \,|\, \tilde{u} = \begin{pmatrix} u(x) \\ u_1 \end{pmatrix} \in H, u(x), u'(x) \in AC\left(\left[-1,\xi_1\right) \cup \left(\xi_1,\xi_2\right) \cup \left(\xi_2,1\right]\right), \\ u(\xi_i \pm 0) = \lim\limits_{x\to\xi_i \pm 0} u(x), u'(\xi_i \pm 0) = \lim\limits_{x\to\xi_i \pm 0} u'(x), (i = 1,2), \\ \ell(u) \in L_2\left[-1,1\right], L_1 u = L_3 u = L_4 u = L_5 u = L_6 u = 0, \\ u_1 = \tilde{R}_1(u), \end{array} \right\}, \tag{9}$$

where $AC\left([a,b]\right)$ is the space of all absolutely continuous functions on the interval $[a,b]$. Hence we can interprete boundary value transmission problem (1)–(7) in H as

$$A\tilde{u} = \lambda\tilde{u}, \tag{10}$$

where $\tilde{u} = \begin{pmatrix} u(x) \\ \tilde{R}_1(u) \end{pmatrix} \in H.$

It is clearly verified that the eigenvalues of A coincide with the eigenvalues of problem (1)–(7) (see Lemma 1.4 in [15]). Also, there exists a correspondence between the eigenfunctions:

$$\tilde{u}_k(x) \leftrightarrow \begin{pmatrix} u_k(x) \\ \tilde{R}_1(u_k) \end{pmatrix}.$$

The operator A is symmetric for $\rho > 0$ (see Theorem 1 in [16]).

2. Main Results

Lemma 1. *The domain $D(A)$ of the operator A is dense in the space H.*

Proof. Let us use the same method as in [1]. Suppose that $\tilde{f} \in H$ is orthogonal to all $\tilde{g} \in D(A)$ with respect to the inner product (8), where $\tilde{f} = \begin{pmatrix} f(x) \\ f_1 \end{pmatrix}$, $\tilde{g} = \begin{pmatrix} g(x) \\ g_1 \end{pmatrix}$. Denote by $\tilde{C}_0^\infty$ the set of functions

$$\Phi(x)=\begin{cases}\phi_1(x), x\in[-1,\xi_1),\\ \phi_2(x), x\in(\xi_1,\xi_2),\\ \phi_3(x), x\in(\xi_2,1],\end{cases}$$

where $\phi_1(x)\in C_0^\infty[-1,\xi_1)$, $\phi_2(x)\in C_0^\infty(\xi_1,\xi_2), \phi_3(x)\in C_0^\infty(\xi_2,1]$. Since $\tilde{C}_0^\infty\oplus 0\subset D(A)$ $(0\in\mathbb{C})$, any $\tilde{u}=\begin{pmatrix}u(x)\\0\end{pmatrix}\in\tilde{C}_0^\infty\oplus 0$ is orthogonal to $\tilde{f}$, namely,

$$(\tilde{u},\tilde{v})=\int_{-1}^{\xi_1}u(x)\overline{v(x)}dx+\delta^2\int_{\xi_1}^{\xi_2}u(x)\overline{v(x)}dx+\delta^2\gamma^2\int_{\xi_2}^{1}u(x)\overline{v(x)}dx+\frac{\delta^2\gamma^2}{\rho}u_1\overline{v_1}=(f,u)_1,$$

where $(,)_1$ denotes the inner product in $L_2[-1,1]$. This implies that $f(x)$ is orthogonal to $\tilde{C}_0^\infty$ and $(f,u)_1=0$. Hence,

$$(\tilde{f},\tilde{g})=\frac{\delta^2\gamma^2}{\rho}f_1\overline{g_1}=0.$$

Therefore, $f_1=0$ since $g_1=\tilde{R}_1(g)$ can be chosen arbitrary. So $\tilde{f}=\begin{pmatrix}0\\0\end{pmatrix}$. Therefore, $D(A)$ is dense in H. This completes the proof of Lemma 1.

Lemma 2. *The operator A is selfadjoint.*

Proof. We know from Lemma 1 that the operator A is dense in the space H. Further, since A is the symmetric operator then, it is sufficient to show that the deficiency spaces are the null spaces and hence $A=A^*$ (where A^* is the adjoint space of A). Now we prove that the inverse of $(A-\lambda I)$ exists. If $Au(x)=\lambda u(x)$,

$$(\overline{\lambda}-\lambda)\langle u,u\rangle_H=\langle u,\lambda u\rangle_H-\langle\lambda u,u\rangle_H=\langle u,Au\rangle_H-\langle Au,u\rangle_H=0.$$

Since $\lambda\notin\mathbb{R}$, we have $\overline{\lambda}-\lambda\neq 0$. Therefore, $\langle u,u\rangle_H=0$, that is $u=0$.

Then $R(\lambda;A):=(A-\lambda I)^{-1}$, the resolvent operator of A exists.

Take $\lambda=\pm i$. The domains of $(A-iI)^{-1}$ and $(A+iI)^{-1}$ are exactly H. Consequently, the ranges of $(A-iI)$ and $(A+iI)$ are also H. Hence the deficiency spaces of A are

$$N_{-i}:=N(A^*+iI)=R(A-iI)^\perp=H^\perp=\{0\},$$
$$N_i:=N(A^*-iI)=R(A+iI)^\perp=H^\perp=\{0\},$$

therefore A is self-adjoint. This completes the proof of Lemma 2.

Theorem 3. *The eigenfunctions of the operator A form an orthonormal basis in the space $H=L_2[-1,1]\oplus\mathbb{C}$.*

Proof. The operator A has countably many eigenvalues $\{\lambda_n\}_{n=1}^\infty$, which have the asymptotic form [16]:

$$\lambda_n=\frac{\pi(n-1)}{2}+O(\frac{1}{n}), n\to\infty.$$

Then, for any number λ, which is not an eigenvalue, and for an arbitary $\tilde{f}\in H$, we can find an element $\tilde{u}\in D(A)$ satisfying the condition $(A-\lambda I)\tilde{u}=\tilde{f}$. Therefore, the operator $(A-\lambda I)$ is invertible except for the isolated eigenvalues. Without loss of generality we assume that the point $\lambda=0$ is not an eigenvalue. Then we obtain that the bounded inverse operator A^{-1} is defined in H. Therefore, the selfadjoint operator A^{-1} has at most countably many eigenvalues, each of which converges to zero at the infinity. Hence, the selfadjoint operator A^{-1} is compact. Applying the Hilbert-Schmidt theorem to this operator we obtain that the eigenfunctions of the operator A form an orthonormal basis in H. This completes the proof of Theorem 1.

Now we consider the case $\rho < 0$. We assume that the operator A is defined by formula (9) on the domain $D(A)$. In the space $H = L_2 \oplus \mathbb{C}$, for $\tilde{u}, \tilde{v} \in H$, the scalar product is defined by the formula

$$(\tilde{u}, \tilde{v}) = \int_{-1}^{\xi_1} u(x)\overline{v(x)}dx + \delta^2 \int_{\xi_1}^{\xi_2} u(x)\overline{v(x)}dx + \delta^2\gamma^2 \int_{\xi_2}^{1} u(x)\overline{v(x)}dx - \frac{\delta^2\gamma^2}{\rho} u_1\overline{v_1}. \tag{11}$$

In this case, the operator A is not selfadjoint in the space H. Therefore, we introduce the operator J as

$$J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

where I is the identity operator in H. The operator J is selfadjoint and invertible.

In this case, boundary value problem (1)–(7) is equivalent to the eigenvalue problem for the operator pencil

$$(B - \lambda J)\tilde{u} = 0 \tag{12}$$

in the space H such that $B = JA$. We obtain that (10) is equivalent to (12).

Lemma 4. *The operator A is $J-$ selfadjoint in the Hilbert space H.*

Proof. Similarly to Lemma 1, we can show that the domain $D(A)$ is dense in the space H. From (11) and (12) applying two times integration by parts, we obtain that $(B\tilde{u}, \tilde{u})$ is real. Hence, the operator B is symmetric. Therefore, the operator A is $J-$ symmetric in the space H. Since A is $J-$ symmetric densely operator, then, similarly to the proof of Lemma 2, it can be shown that the operator JA is selfadjoint. This completes the proof of Lemma 4.

Corollary 5. One element of the system $\{u_n\}_0^\infty$ can be eliminated such that the remaining elements form a complete and minimal system in the space $L_2[-1,1]$.

Proof. By Theorem 1, the system of eigenfunctions

$$\tilde{u}_n(x) = \begin{Bmatrix} u_n(x) \\ u_1 \end{Bmatrix},$$

$(u_1 \in \mathbb{C})$ of the operator A forms an orthonormal basis in H. Hence, the system of the eigenfunctions $\{\tilde{u}_n(x)\}_1^\infty$ is complete and minimal in the space H. Therefore, of course, $co\dim P = 1$, then by Lemma 2.1 in [15], the system $\{P\tilde{u}_n(x)\} = \{u_n(x)\}$ whose one element is omitted forms a complete and minimal system in $P(H) = L_2[-1,1]$. Hence, the eigenfunctions $\{u_n(x)\}_0^\infty$ ($n \neq n_0$, n_0 is an arbitrary nonnegative integer) of boundary value problem (1)–(7) form complete and minimal system in $L_2[-1,1]$. This completes the proof of Corollary 1.

Theorem 6. *The eigenfunctions of the operator A form a Riesz basis in the Hilbert space H.*

Proof. Since the operator J is a bounded operator, then using Theorem 1, it can be shown that the operator $B = JA$ is invertible because of the fact that the selfadjoint operator B^{-1} is compact. The selfadjoint operator B^{-1} has at most countably many eigenvalues which converge to zero at infinity. Hence the operator B^{-1} is compact. Then applying Theorem 2.12 in Section IV of [3] to the operator B, we obtain that the eigenfunctions of the $J-$ selfadjoint operator A form a Riesz basis in the space $H = L_2 \oplus \mathbb{C}$. This completes the proof of Theorem 2.

3. Results and Discussion

The paper is devoted to one class of the Sturm–Liouville operators with the eigenparameter-dependent boundary conditions and the transmission conditions. A new operator A associated with the problem is established, some spectral properties of this operator is examined in an appropriate space H and basisness of its eigenfunctions is discussed.

The Sturmian theory is an important tool in solving many problems of mathematical physics. Usually, the eigenvalue parameter appears only linearly in the differential equation of the classic Sturm–Liouville problems. However, in this study the eigenvalue parameter appears both in the differential equation and boundary condition. Moreover, two transmission conditions at two points are added. Therefore, the problem is different from the classic Sturm–Liouville problems and it has novelity.

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О БАЗИСНОМ СВОЙСТВЕ КОРНЕВЫХ ФУНКЦИЙ ОДНОГО КЛАССА ДИФФЕРЕНЦИАЛЬНЫХ ОПЕРАТОРОВ ВТОРОГО ПОРЯДКА

В. Ала, Х.Р. Мамедов
Мерсинский университет, Мерсин, Турция
E-mail: volkanala@mersin.edu.tr, hanlar@mersin.edu.tr

Хорошо известно, что теория Штурма является важным инструментом решения широкого класса задач математической физики. Как правило, в классических задачах Штурма–Лиувилля собственные значения линейно входят только в дифференциальное уравнение. Однако в математической физике встречаются задачи, в которых собственные числа появляются не только в дифференциальном уравнении, но и в граничных условиях.

В этой статье мы рассматриваем задачу Штурма–Лиувилля, собственные значения которой входят в уравнение, присутствуют в граничных условиях и дополнительно должны быть согласованы с условиями прохождения решения через две фиксированные точки разрыва.

Целью данной работы является исследование полноты, минимальности и базисных свойств корневых функций рассматриваемой краевой задачи.

Ключевые слова: собственные функции; ортонормированный базис; базис Рисса; полнота.

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